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October 2008

Abstract

This article proposes a non-parametric portfolio selection criterion for the static asset allocation problem in a robust higher-moment framework. Adopting the Shortage Function approach, we generalize the multi-objective optimization technique in a four-dimensional space using L-moments, and focus on various illustrations of a four-dimensional set of the first four L-moment primal efficient portfolios. Our empirical findings, using a large European stock database, mainly rediscover the earlier works by Jean (1973) and Ingersoll (1975), regarding the shape of the extended higher-order moment efficient frontier, and confirm the seminal prediction by Levy and Markowitz (1979) about the accuracy of the mean-variance criterion.

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Abstract

This article proposes a non-parametric portfolio selection criterion for the static asset allocation problem in a robust higher-moment framework. Adopting the Shortage Function approach, we generalize the multi-objective optimization technique in a four-dimensional space using L-moments, and focus on various illustrations of a four-dimensional set of the first four L-moment primal efficient portfolios. Our empirical findings, using a large European stock database, mainly rediscover the earlier works by Jean (1973) and Ingersoll (1975), regarding the shape of the extended higher-order moment efficient frontier, and confirm the seminal prediction by Levy and Markowitz (1979) about the accuracy of the mean-variance criterion.

Since the first mention of higher-order moments than the variance of returns by Marschak (1938) and Hicks (1939), it is now generally accepted by the financial community that investors generally exhibit preferences for positively skewed and light-tailed asset return distributions (see, for instance, Beedles and Simkowitz (1978), Dittmar (2002), Jurczenko and Maillet (2006a), and Mitton and Vorkink (2007)). Developments with higher moments followed since the origin three main (complementary) directions in finance: a tentative integration in the von Neumann-Morgenstern utility function of a rational investor (*Cf.* Arrow (1964) and Pratt (1964)) of higher-order moments of returns by Arditti (1967), Samuelson (1970) and Tsiang (1972); an attempt to generalize the Markowitz (1952) efficient frontier to incorporate the effect of higher moments on optimal asset allocations by Jean (1971 and 1973), Arditti and Levy (1972), Ingersoll (1975) and Schweser (1978); and a first partial explicit modelling of returns by Rubinstein (1973), and Kraus and Litzenberger (1976), through extensions of the CAPM by Sharpe (1964). True departures

from Gaussianity may indeed affect the optimal allocation of assets (see Jondeau and Rockinger (2003b and 2006a)) and the mean-variance portfolio selection *criterion* proposed by Markowitz (1952) is *a priori* somehow inadequate for some risky assets whose characteristics are very special (see Jondeau and Rockinger (2006a)). In such a context, different multi-moment approaches have been proposed in the financial literature to incorporate higher-order moment preferences into asset allocation problems¹, in order to characterize generalized geometric efficient frontiers (see Athayde and Flôres (2002 and 2003)); but all suffer more or less from the traditional drawbacks of algebraic moments. First, higher-order moments do not always exist (see Embrechts, Kluepelberg and Mikosh (1997), and Jondeau and Rockinger (2003a and 2003b)), and even when they do, such moments do not always uniquely define a probability distribution, so that two distinct distributions can have the same sequence of moments (see Heyde (1963) for the example of the Log-normal density). Secondly, conventional moments tend to be very sensitive to a few extreme observations (see Hampel, Ronchetti, Rousseeuw and Stahel (2005)). The asymptotic efficiency of the empirical moments is also rather poor, especially for distributions with fat tails. This last property is an immediate consequence of the fact that the asymptotic variances of these estimators are mainly determined by higher-order moments, which tend to be rather large, or even unbounded, for heavy-tailed distributions.

The objective of this paper is to overcome the limits of traditional multi-moment asset allocation models by using an alternative set of statistics in traditional optimization programs, namely L-moments. Recent attempts for modelling distributions in a multivariate framework are indeed built on the concept of order-statistics, for calibrating a Bernstein *Copula* in Baker (2008) or for defining extreme co-movements using L-moments in Serfling and Xiao (2007). The latter, which are linear functions of the expectations of order statistics, were introduced under this name by Sillitto (1951) and comprehensively reviewed by Hosking (1989). As so-called U-statistics (see Hoeffding (1948)), L-moments offer one main advantage over Conventional moments (denoted herein C-moments as in Ulrych, Velis, Woodbury and Sacchi (2000), and Chu and Salmon (2008)). Their empirical counterparts are less sensitive to the effects of sampling variability, since they are linear functions of the ordered data, and are therefore shown to provide more robust estimators

¹See Athayde and Flôres (2002 and 2006), Briec, Kerstens and Jokung (2007), Jurczenko and Maillet (2006b), and Jurczenko, Maillet and Merlin (2006) for the primal approaches of the mean-variance-skewness-*kurtosis* portfolio decision problems; and Simaan (1993), Gamba and Rossi (1998a and 1998b), Jurczenko and Maillet (2001 and 2006a), and Jondeau and Rockinger (2006a) for the dual approaches of the higher-order moment asset allocation problems.

of higher moments than the corresponding sample C-moments (Sankarasubramanian and Srinivasan (1999)). More precisely, L-moments are defined as certain linear functions of the Probability Weighted Moments (Greenwood, Landwehr, Matalas and Wallis (1979)) and can characterize a wider range of distributions compared to C-moments. Indeed, they exist whenever the mean of the distribution does, even though some C-moments do not (which is very likely to be the case in finance). As we will see later on, they are also particularly well adapted for addressing some specific concerns in the field of finance. And the beauty is that they are easy and fast(er) to compute, besides being reliable estimators of characteristic shape parameters of general distributions.

Because of their proven advantages, L-moments have already found wide applications in various fields such as meteorology, hydrology, geophysics and regional analysis (see Hosking and Wallis (1997)) - where large deviations really matter, namely for instance when studying extreme floods or low flows (see among the main references: Hosking and Wallis (1987), Ben-Zvi and Azmon (1997), Wang (1997), Bayazit and Önöz (2002), Moissello (2007), and Shao, Chen and Zhang (2008)), rainfall extremes (see Guttman, Hosking and Wallis (1993), Lee and Maeng (2003), and Parida and Moalafhi (2008)), raindrop sizes (see Kliche, Smith and Johnson (2008)), velocity of gale force winds (see Pandey, Gelder and Vrijling (2001), Whalen, Savage and Jeong (2004), and Modarres (2008)) or the measurement of earthquake intensities (Thompson, Baise and Vogel (2007)). Lately, they have also found an interest in finance, first, as *Monsieur Jourdain* by Molière without noticing it, when using a special case of an L-moment which is the Gini coefficient (see Gini (1912) and below) as a substitute to the volatility in asset pricing models (see Shalit and Yitzhaki (1989), Okunev (1990), and Benson, Faff and Pope (2003)); then, secondly, for more general purposes: for fitting return distributions (Hosking, Bonti and Siegel (2000), Carrillo, Hernández and Seco (2006a), and Karvanen (2006)) and the rate of profit densities (Wells (2007)), the design of a GMM-type Goodness-of-Fit test (see Chu and Salmon (2008)), risk modelling purposes (see Martins-Filho and Yao (2006), Tolikas and Brown (2006), Tolikas, Koulakiotis and Brown (2007), Gouriéroux and Jasiak (2008), and Tolikas (2008)), calibrating extreme return distributions (Gettinby, Sinclair, Power and Brown (2006), French (2008), and Tolikas and Gettinby (2008)) or rogue-volatility densities (Maillet and Médecin (2008), and Maillet Médecin and Michel (2008)), and very recently for defining a new set of measures of performance for hedge funds (see Darolles, Gouriéroux and Jasiak (2008)).

Thanks to the so-called shortage function technique and relying on robust L-statistics, we thus generalize in this article the traditional mean-variance-skewness-*kurtosis* efficient

frontier in the four L-moment space, proposing a new and fast formulation of higher-order (L-)comoments of efficient portfolios. The shortage function of Luenberger (1995) was indeed first applied to the portfolio performance evaluation in the traditional mean-variance framework by Morey and Morey (1999), then developed by Briec, Kerstens and Lesourd (2004), and recently extended to multi-horizon performance appraisals (Briec and Kerstens (2009)). In brief, the shortage function rates the performance of any portfolio by measuring a distance between the coordinates of this specific portfolio and those of its radial projection onto the multi-moment efficient frontier. Based on this distance definition, the so-called Goal Attainment Method enables us to solve the multiple conflicting and competing allocation objectives, without assuming a detailed knowledge of the preference parameters of the indirect investor's utility function.

After a theoretical presentation of L-moments and of the shortage function approach in a portfolio selection context, we propose to rewrite the multi-moment optimization program of the investor within a new compact notation, using both C-moments and L-moments, then derive the four L-moment efficient set and provide various illustrations using a universe of 162 European stocks. Our empirical results regarding links between moments of efficient portfolios and the various shapes of the higher-order moment efficient frontier - from (*pseudo*-)parabolas to (deformed) cones, confirm the earlier findings by Jean (1973) and Ingersoll (1975) presenting the first three-dimensional representation of the efficient set. They are also consistent with more recent evidences provided in other frameworks (see for instance, Athayde and Flôres (2004), Jurczenko, Maillet and Merlin (2006), Maringer and Parpas (2008)). However, it is worth noting that our attempt to evaluate the cost of not using higher-order moments is still not conclusive in a traditional "mixed" utility setting, since differences between optimal asset allocation implied utilities are found to be marginal. In other words, the mean-variance *criterion* - corner stone of the Modern Portfolio Theory of de Finetti² (1940) and Markowitz (1952) - is shown to be rather accurate as predicted, among others, by Levy and Markowitz (1979) and Kroll, Levy and Markowitz (1984). As mentioned by Jondeau and Rockinger (2006a), the use of higher-order moments, in a the traditional expected utility framework and in a restrictive "mixed" utility setting (in which sensitivities only depends upon the first moment), may thus only prove their efficiency either if the underlying assets are largely non-Gaussian (in some specific sense) or if the representative investor exhibits some very peculiar features regarding her prudence and temperance characteristics. Nevertheless, we also suggest that

²See Markowitz (2006), Pressacco and Serafini (2007) and Barone (2008).

complementary controls of higher-order moments of some traditional low-dispersion return efficient portfolios (such as the Global Minimum-Variance Portfolio) may be of interest for the investors.

The remainder of the article is organized as follows. In section 2, we formally present the L-moments, briefly recall some of their main properties, and illustrate their computations on a long record of stock index quotes. In section 3, we precisely define in a new notation higher-order moments of returns on portfolios and describe how the optimal portfolio selection L-moment program can be solved in a shortage function framework. In section 4, we present the data and discuss the results of the various optimal asset allocations. Section 5 concludes. The Appendices are dedicated to proofs, some technical details, Tables and Figures.

I. Robust Higher-order Moments for Portfolio Selection

Introduced by Sillitto (1951) and popularized by Hosking (1989), L-moments can be interpreted, like C-moments, as simple descriptors of the shape of a general distribution, albeit offering a number of advantages.

First, all population (higher) L-moments exist and uniquely determine a probability distribution, provided that the mean exists (see Chan (1967), and Arnold and Meeden (1975)). In this case, a distribution can always be specified by its L-moments, even if some of its higher-order C-moments do not exist. Furthermore, this specification is always unique. For the standard errors of L-moments to be finite, it is also only required that the distribution has a finite variance; no condition on higher-order moments is necessary (Hosking (1990)). Moreover, although moment ratios can be arbitrarily large, sample moment ratios have algebraic bounds (see Dalén (1987)) and sample L-moment ratios can take any values that the corresponding population quantities can (Hosking (1990)). Motivated by the sampling properties of L-statistics, Hosking and Wallis (1987) thus advocate that L-moments provide a better approximation of the unknown parent distribution than C-moments. They also provide reliable estimators for Extreme Value densities and have been widely used in fields where exceptions are at stake (see Hosking (1990)).

Secondly, L-moments exhibit some specifically interesting features for financial applications. Since a (complete) set of L-moments determine a unique density, the so-called Hamburger problem (see Jondeau and Rockinger (2003a) and Jurczenko and Maillet (2006a))

- when C-moments lead to several laws - is limited. Since they always exist, the problem of working with non-defined quantities such as higher C-moments is avoided. L-moments are also coherent shape measures of risk (see Artzner, Delbaen, Eber and Heath (1999)), since they are translation and scale invariants (Serfling and Xiao (2007), and Gouriéroux and Jasiak (2008)). They, furthermore, allow us a clearer focus on a specific part of the distribution, thus avoiding confusion between the center and the extreme parts of the distribution as in the traditional case (see Haas (2007)). Finally, the sample estimates of L-moments are more robust to data outliers (Vogel and Fennessey (1993)) - since they are only linearly influenced by large deviations (see Hosking (1990)) - and more efficient than C-moments (see Sankarasubramanian and Srinivasan (1999), and Carrillo, Hernández and Seco (2006b)), especially within a Generalized Method of (L-)Moment context (see Gettinby, Sinclair, Power and Brown (2006), Chu and Salmon (2008), and also Gouriéroux and Jasiak (2008)).

After having briefly recalled the different analytical representations of the univariate population L-moments³ - we mainly refer here to the work by Hosking and Wallis (1997), we will present in the following sub-section their sample unbiased estimator counterparts and shall illustrate the four first L-moment estimates on a long sample of one century of daily quotes of the Dow Jones Index.

A. Population L-comoments

Population L-moments are defined as certain linear functions of the expectations of the order statistics from the population distribution of the underlying random variable. Let us start with some basic notations and definitions.

Let $\{X_t\}$, with $t = [1, \dots, T]$, be a conceptual random sample of size T drawn from a continuous probability distribution $F(\cdot)$ of a real-valued random variable X , with $T \in \mathbb{N}^*$; $Q(u) = F^{-1}(u)$, for $u \in]0, 1[$, a quantile function, and $X_{[1:T]} \leq X_{[2:T]} \leq \dots \leq X_{[T:T]}$ denoting the corresponding order statistics. Then the k -th population univariate L-moment (using the L-functional representation) is defined, $\forall k \in \mathbb{N}^*$ and $k < T$, as

³Other variants of L-moments, called TL-moments (encompassing the PL-moments, the LH-moments and the LL-moments), LQ-moments (see Maillet and Médecin (2008), for a comprehensive review and an application to extreme volatilities), as well as the LSD-moments (SD-PWM - see Haktanir (1997), and Whalen, Savage and Jeong (2004)), have already been used in extreme studies, since they are proven to be even less sensitive to outliers. However, to be as close as possible to the portfolio value as perceived by the investors, we will stick, in this article, with the traditional simple L-moments in our portfolio context.

(see, for instance, Hosking (1990)):

$$\begin{aligned}\lambda_k(X) &= \sum_{j=0}^{k-1} (-1)^j \{ [k(k-j-1)!j!]^{-1} [(k-1)!] \} \times E(X_{[k-j:k]}) \\ &= \int_0^1 Q(u) P_{k-1}^*(u) du,\end{aligned}\tag{1}$$

with:

$$\begin{cases} E(X_{[r:k]}) &= \{ [(r-1)!(k-r)!]^{-1} (k!) \} \times \int_0^1 Q(u) u^{r-1} (1-u)^{k-r} du \\ P_k^*(u) &= \sum_{r=0}^k p_{k,r}^* u^r \\ p_{k,r}^* &= (-1)^{k-r} \{ [(r!)^2 (k-r)!]^{-1} (k+r)! \}, \end{cases}$$

where $\lambda_k(\cdot)$ is the L-moment of order k , $r = [1, \dots, k]$, $0 < u < 1$, $E(\cdot)$ is the expectation operator and $p_{k,r}^*$ corresponds to the r -th coefficient of the shifted orthogonal Legendre polynomial of degree k denoted $P_k^*(\cdot)$ and defined as $P_k^*(u) = P_k(2u-1)$ where $P_k(\cdot)$ is the traditional Legendre polynomial of degree k .

Thus, the shifted orthogonal Legendre polynomial satisfies, $\forall (k, s) \in IN^{*2}$:

$$\begin{cases} \int_0^1 P_r^*(u) P_s^*(u) du = (2r-1)^{-1} & \text{if } r = s \\ = 0 & \text{if } r \neq s, \end{cases}\tag{2}$$

with $P_0^*(u) = 1$.

In particular, the first four population L-moments are:

$$\begin{cases} \lambda_1(X) = E(X_{[1:1]}) \\ \lambda_2(X) = (2)^{-1} E(X_{[2:2]} - X_{[1:2]}) \\ \lambda_3(X) = (3)^{-1} E[(X_{[3:3]} - X_{[2:3]}) - (X_{[2:3]} - X_{[1:3]})] \\ \lambda_4(X) = (4)^{-1} E[(X_{[4:4]} - X_{[1:4]}) - 3(X_{[3:4]} - X_{[2:4]})]. \end{cases}\tag{3}$$

Knowing relation (1), they also satisfy the following equalities:

$$\begin{cases} \lambda_1(X) = \int_0^1 Q(u) du \\ \lambda_2(X) = \int_0^1 Q(u) (2u-1) du \\ \lambda_3(X) = \int_0^1 Q(u) (6u^2 - 6u + 1) du \\ \lambda_4(X) = \int_0^1 Q(u) (20u^3 - 30u^2 + 12u - 1) du. \end{cases}\tag{4}$$

Note that $\lambda_1(\cdot)$, $\lambda_2(\cdot)$, $\lambda_3(\cdot)$ and $\lambda_4(\cdot)$ are population measures of location, scale and shape, strictly analogous to the corresponding traditional central moments. The first L-moment is the mean of the population distribution and the second L-moment, defined

in terms of a conceptual random sample of size 2, is a measure of the typical spread of the random variable X , being half the expected value of Gini's Mean Difference (see Gini (1912)), which has some desirable properties in finance when replacing the traditional variability measure (see Yitzhaki (2003)). In short, the third L-moment simply represents the difference between the upper tail and the lower tail, and hence measures the asymmetric shape of the population distribution from which the conceptual random sample has been drawn. Similarly, the fourth L-moment measures the *kurtosis* of the probability distribution function, and can be expressed as a (rescaled) difference between the typical spread in the tails and the typical spread in the center.

While Equation (1) is the classical L-functional's representation for the population L-moments, there exist several other representations for $\lambda_k(\cdot)$ that prove to be useful for financial applications (see below). For example, using the definition of the Probability Weighted Moments, we can also express Equation (1) as a linear function of the Probability Weighted Moments, that is (see Hosking (1989)):

$$\lambda_k(X) = \sum_{r=1}^k p_{k-1, r-1}^* \beta_{r-1}(X), \quad (5)$$

with:

$$\beta_{r-1}(X) = r^{-1} E(X_{[r:r]}) = \int_0^1 Q(u) u^{r-1} du,$$

where $k \in \mathbb{N}^*$, $\beta_r(\cdot)$ are the Probability Weighted Moments of order r , with $r = [2, \dots, k]$, and $p_{k-1, r-1}^*$ is the $(r-1)$ -th coefficient of the shifted Legendre polynomial of degree $(k-1)$ defined in Equation (1).

Using the definition and orthogonal property of the shifted Legendre polynomials, the k -th population L-moment can also be represented as the covariance between the random variable X and its distribution function denoted $F(\cdot)$, that is (see Serfling and Xiao (2007)):

$$\lambda_k(X) = \begin{cases} E(X) & \text{if } k = 1 \\ Cov\{X, P_{k-1}^*[F(X)]\} & \text{if } k \neq 1, \end{cases} \quad (6)$$

with:

$$E\{P_k^*[F(X)]\} = \int_0^1 P_k^*(u) du = 0,$$

where $P_0^*(u) = 1$, with $k \in \mathbb{N}^*$, $P_k^*(\cdot)$ the shifted orthogonal Legendre polynomial of degree k defined as previously, and $Cov(\cdot, \cdot)$ the covariance operator.

Since $E[F(X)] = 1/2$, relation (6) allows us to obtain alternative expressions for the first four population L-moments such as:

$$\begin{cases} \lambda_1(X) = E(X) \\ \lambda_2(X) = 2 E \{ [X - E(X)] \times [F(X) - E[F(X)]] \} \\ \lambda_3(X) = 6 E \{ [X - E(X)] \times [F(X) - E[F(X)]]^2 \} \\ \lambda_4(X) = 20 E \{ [X - E(X)] \times [F(X) - E[F(X)]]^3 \} - 3 (2)^{-1} \lambda_2(X), \end{cases} \quad (7)$$

where $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ represent respectively the (L-)mean and the L-variance, and $\lambda_3(\cdot)$ and $\lambda_4(\cdot)$ correspond to the unscaled L-skewness and L-kurtosis of the population distribution $F(\cdot)$.

The population L-moments presented in Equation (7) are defined for a given probability distribution, but, in practice, they are directly estimated with some uncertainty from a finite sample draw, corresponding to an unknown distribution. The following sub-section is devoted to a brief presentation of the (original) sample counterparts of the population L-moments.

B. Sample Portfolio Return L-moments

Following Hosking (1989), sample L-moments can be estimated in a straightforward manner by estimating the empirical Probability Weighted Moments, denoted $\hat{\beta}_r(\cdot)$, in conjunction with their L-statistic representations in (5).

Indeed, let $X_{[1:T]} \leq X_{[2:T]} \leq \dots \leq X_{[T:T]}$ be again the order statistics of a random sample $\{X_t\}$ of size T , drawn from a continuous non-degenerated probability distribution $F(\cdot)$ of a real-valued random variable X , with $t = [1, \dots, T]$ and $T \in \mathbb{N}^*$. The r -th Probability Weighted Moment estimator is given, $\forall r \in \mathbb{N}^*$ and $1 < r < T$, by:

$$\hat{\beta}_r(X) = (T)^{-1} \times \sum_{t=1}^T \left\{ \prod_{j=1}^r \left[\frac{(t-j)}{(T-j)} \right] X_{[t:T]} \right\}, \quad (8)$$

where $\hat{\beta}_r(\cdot)$ is the unbiased estimator of $\beta_r(\cdot)$.

Thus, for a sample of size T , the k -th sample L-moment is defined as:

$$\hat{\lambda}_k(X) = \sum_{r=1}^k p_{k-1, r-1}^* \hat{\beta}_{r-1}(X), \quad (9)$$

where $\hat{\lambda}_k(X)$ is the k -th sample L-moment corresponding to the population L-moment $\lambda_k(\cdot)$ of a real-valued random variable X , $(r \times k \times T) \in (\mathbb{N}^*)^3$ with $r < k < T$, $p_{r-1, k-1}^*$

is the $(r - 1)$ -th coefficient of the shifted Legendre polynomial of degree $(k - 1)$ defined as in Equation (1).

The L-moment estimator (9) is an unbiased L-statistic estimator of the population L-moment. In particular, in a time-series context, the first four sample L-moments from (9) are given by (with previous notations):

$$\left\{ \begin{array}{l} \hat{\lambda}_1(X) = (T)^{-1} \sum_{t=1}^T X_{[t:T]} \\ \hat{\lambda}_2(X) = [T(T-1)]^{-1} \sum_{t=1}^T (2t-1-T) X_{[t:T]} \\ \hat{\lambda}_3(X) = [T(T-1)(T-2)]^{-1} \sum_{t=1}^T [6(t-1)t - 6(t-1)T \\ \quad + (T-1)(T-2)] X_{[t:T]} \\ \hat{\lambda}_4(X) = [T(T-1)(T-2)(T-3)]^{-1} \\ \quad \times \sum_{t=1}^T [20(t-1)(t-2)(t-3) - 30(t-1)(t-2)(T-3) \\ \quad + 12(t-1)(T-2)(T-3) - (T-1)(T-2)(T-3)] X_{[t:T]}. \end{array} \right. \quad (10)$$

Under the second central moment existence condition, the standard theory for U-statistics and L-statistics states that the (first) k -th sample L-moments are asymptotically jointly normally distributed, with a similar result for the vector of the sample L-moment ratios (see, for instance, Hoeffding (1948)).

Figure 1 illustrates the first (rescaled) L-moment estimations against the traditional sample moments, calculated on a long sample of one century of daily quotes of the Dow Jones Index. The set of figures on the left part (Figure 1) corresponds to time-evolutions of the second, third and fourth moments, recursively computed since the 1st of January 1900, whilst the set of right figures represents the one-year rolling window first moment non-parametric empirical densities. As expected, it is clear from these figures that sample L-moments are far more stable than C-moment estimates. Moreover, densities of L-moments are more concentrated around a unique mode of L-moment values and exhibit fewer extreme values, indicating faster decreasing tails. This visually confirms that higher-order L-moments are less prone to the influence of outliers, and thus may be seen as more accurate.

- Please, Insert Figure 1 somewhere here -

The main properties of population L-moments and their corresponding sample quantities now stated and their empirical robustness illustrated, we generalize in the next section,

by using the shortage function approach of Luenberger (1995), the mean-variance efficient frontier in the first four L-moment space.

II. Portfolio Selection with Higher-order Moments

In the context of portfolio selection, the aim of the investors is to determine their asset allocation in order to maximize their utility function. We refer here to a general class of utility functions exhibiting a “mixed” risk aversion respecting the fourth-order stochastic dominance criterion (see Caballé and Pomansky (1996)), which alternate the signs of partial derivatives. In such a setting, we thus consider an exact (or accurate approximative) fourth-order Taylor expansion of a general utility function (see for details Jurzcenko and Maillet (2006a), and Garlappi and Skoulakis (2008)) with a strictly (monotone) increasing first derivative representative of the preference of non-satiable individuals, a strictly decreasing second derivative for risk-averse agents, a strictly increasing third derivative for prudent investors, and a strictly decreasing fourth derivative for temperate behaviors. More precisely (see Appendix 1 for a few illustrations with some usual utility functions), the expected utility of the random return on a portfolio p (denoted R_p) held by a rational investor, can be represented by an indirect utility function, denoted $V(\cdot)$, successively, concave and increasing with the expected return - denoted $E(R_p)$, concave and decreasing with the variance - reading $\sigma^2(R_p)$, concave and increasing with the skewness - written $m^3(R_p)$, and, concave and decreasing with the *kurtosis*⁴ - defined by $\kappa^4(R_p)$. Such an expected utility function can be written in a general form as:

$$E[U(R_p)] = V[E(R_p), \sigma^2(R_p), m^3(R_p), \kappa^4(R_p)], \quad (11)$$

with:

$$V_1 = \frac{\partial V(\cdot)}{\partial E(R_p)} > 0, V_2 = \frac{\partial V(\cdot)}{\partial \sigma^2(R_p)} < 0, V_3 = \frac{\partial V(\cdot)}{\partial m^3(R_p)} > 0 \text{ and } V_4 = \frac{\partial V(\cdot)}{\partial \kappa^4(R_p)} < 0,$$

where $R_p = W/W_0 - 1$ is the (random) return on the portfolio p held by the investors, with W_0 their initial wealth (being equal to unity for the sake of simplicity), W their random final wealth, and $V(\cdot)$ a general (non-)linear indirect utility function whose arguments are the first four conventional moments of returns on portfolio p .

⁴Whilst, in general, skewness and *kurtosis* correspond to the standardised third and fourth centered moment, they are used here as the third and fourth centered moment.

The various first derivatives of such a general indirect utility function characterize at the same time both economic agent behavioural assumptions - his rational reaction to increases in downside risk, fear of ruin, will of self-protection and self-insurance (see Chiu (2005 and 2008), Crainich and Eeckhoudt (2008)), and a (simple) transformation of a density function of his return on wealth (see, for instance, Eeckhoudt, Gollier and Schneider (1995)). More precisely, the first derivative with respect to the expected return governs the so-called “greediness” of the investor, the second sensitivity represents his “risk aversion”, whilst the third⁵ and fourth terms characterize respectively the “prudence” (Kimball (1990) and Lajeri-Chaherli (2004)) and “temperance” (Kimball (1992 and 1993), Eeckhoudt, Gollier and Schneider (1995), Menezes and Wang (2005))⁶.

From a theoretical point of view, the link between moments and preferences (Scott and Horwath (1980)) is still under question for at least four main reasons. First, moments are only single statistics that can only be imperfect summaries of the plain return distribution characteristics (see Romano and Siegel (1986))⁷. Depending on the exact transfer (preserving function) of probability weights, we can imagine all sorts of density distortions that basically break the rationale of the investor choice when comparing asset allocations (see Brockett and Kahane (1992) for some explicit examples). Secondly, first moments merely share the same information contained in the return series: the higher the order (and the power of the conventional moment), the more important the focus on the tail and extreme events *ceteris paribus*. Hence, some of them exhibit certain correlations *per construction*. In other words, redundant information⁸ is present in the various comoments (see Galaged-

⁵Some interesting recent works, however, also show that the ratio $[U'''(.)]/U'(.)$ is also link to a risk aversion characteristic of a rational agent, who makes an arbitrage between the first and the third moments (Cf. Crainich and Eeckhoudt (2008)).

⁶Lajeri-Chaherli (2004) extends the expansion to the order five, mentioning in reference the fifth-order risk apportionment “edginess”, whilst Caballé and Pomansky (1996) refine even further the expansion to the n -th order, referring to the “risk aversion of order n ”, as an analogue to the traditional classical absolute risk aversion (see also Eeckhoudt and Schlesinger, (2006)). However, to our knowledge, no more precise label of utility characteristic yet exists for extensions to higher-order moments than the fifth. In the following, we nevertheless restrict our analysis to the four first moments, mainly for the sake of tractability, but also due to some questions regarding the existence of empirical counterparts of higher-order C-moments.

⁷The counterexample they mentioned being that the apparently left-skewed distribution of $x = \{-2, 1, 3\}$ with associated probabilities of $f(x) = \{.4, .5, .1\}$, has a null skewness.

⁸It is straightforward to show, for instance, that some terms in the conventional *cokurtosis* matrix also appear in both covariance and coskewness matrices (see among others Jurczenko and Maillet (2006a), on notations of higher-order co-moments).

era and Maharaj (2008)). Thirdly, the explicit expressions of links between moments and preferences strongly depend upon the precise preference definition (see Haas (2007)) and on the performance of measures of higher-order moments (see Kim and White (2004), for various measures). Fourthly, in a more general prospect theory framework (Kahneman and Tversky (1979)), the rational investor may also further relax the linearity-in-probability property inherent in expected utility theory, by allowing the physical probabilities to be nonlinearly subjectively transformed into “decision weights” (see Kliger and Levy (2008)).

Nevertheless, from a more practical point of view, a wealth of literature (see Jondeau and Rockinger (2006a), Jurczenko and Maillet (2006a), and Briec and Kerstens (2007), for a precise reference list on the subject) points out a realistic positive preference of investors for the highest right asymmetries and the lowest tail-fatnesses; we will take this common sense *hypothesis* as granted in the following, where we simply try to extend the two-moment Markowitz’ analysis in an expected utility framework, only considering higher moments with better properties.

Since we will also only consider “mixed” utility functions, the signs of the sensitivities V_n (partial derivatives), for $n = [1, \dots, 4]$, will alternate. Furthermore, in a portfolio context, one could intuitively expect that the investor cares more about a (positive) expected return than about other characteristics⁹, and, as a result, that the sensitivities decrease with the order of related moments.

In such a framework, the agent’s portfolio general problem can be stated as (with previous notations):

$$\begin{aligned} \text{Max}_{\mathbf{w}'_p} \{E[U(R_p)]\} &= \text{Max}_{\mathbf{w}'_p} \{V[E(R_p), \sigma^2(R_p), m^3(R_p), \kappa^4(R_p)]\} \\ \text{s.t.} \quad &: \mathbf{w}'_p \mathbf{1}_N = 1, \end{aligned} \tag{12}$$

where $\mathbf{1}_N$ is the $(N \times 1)$ unit vector and $V(\cdot)$ a general non-explicated (non-)linear function depending on the four first C-moments of returns on portfolio p that we explicit hereafter.

⁹It is difficult to believe that most rational investors care more about higher-order moments than about the expected return in their asset allocation decisions (which can be the case in a lottery for instance, where the potential big prize may entail a large skewness, allowing them to forget in some sense the likely negative profit of the game). This is the reason why in the following, we further restrict the search for optimal portfolios in regions where the expected returns are positive, and only select, in some representations, portfolios where impacts on the utility functions of moments are ranked according to their order (see Section 4).

A. Higher-order C-comoments of Portfolio Returns

Actually, the mean, variance, skewness and *kurtosis* of portfolio p returns used in Equation (11) are given by, with $(i, j, k, l) = [1, \dots, N]^4$ (and with previous notations):

$$\left\{ \begin{array}{l} E(R_p) = \sum_{i=1}^N w_{pi} E(R_i) \\ \sigma^2(R_p) = E\{[R_p - E(R_p)]^2\} = \sum_{i=1}^N \sum_{j=1}^N w_{pi} w_{pj} \sigma_{ij} \\ m^3(R_p) = E\{[R_p - E(R_p)]^3\} = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N w_{pi} w_{pj} w_{pk} m_{ijk} \\ \kappa^4(R_p) = E\{[R_p - E(R_p)]^4\} = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N w_{pi} w_{pj} w_{pk} w_{pl} \kappa_{ijkl}, \end{array} \right. \quad (13)$$

with:

$$\left\{ \begin{array}{l} \sigma_{ij} = E\{[R_i - E(R_i)][R_j - E(R_j)]\} \\ m_{ijk} = E\{[R_i - E(R_i)][R_j - E(R_j)][R_k - E(R_k)]\} \\ \kappa_{ijkl} = E\{[R_i - E(R_i)][R_j - E(R_j)][R_k - E(R_k)][R_l - E(R_l)]\}, \end{array} \right.$$

where w_{pi} , R_i , σ_{ij} , m_{ijk} and κ_{ijkl} represent, respectively, the weight of the asset i in portfolio p , the return on the asset i , the covariance between the returns on assets i and j , the coskewness between the returns on assets i , j and k , and the *cokurtosis* between the returns on assets i , j , k and l .

These various C-moments of portfolio returns were previously written in a matrix format (see Diacogiannis (1994), Athayde and Florès (2002, 2003, 2004 and 2006), Harvey, Liechty, Liechty, Mueller (2002), Prakash, Chang and Pactwa (2003), Jondeau and Rockinger (2003a, 2003b and 2006a) and Jurczenko, Maillet and Merlin, (2006)) defined as such (with previous notations):

$$\left\{ \begin{array}{l} E(R_p) = \mathbf{w}_p' \mathbf{E} \\ \sigma^2(R_p) = \mathbf{w}_p' \mathbf{\Omega} \mathbf{w}_p \\ m^3(R_p) = \mathbf{w}_p' \times \mathbf{\Sigma} \times (\mathbf{w}_p \otimes \mathbf{w}_p) \\ \kappa^4(R_p) = \mathbf{w}_p' \times \mathbf{\Gamma} \times (\mathbf{w}_p \otimes \mathbf{w}_p \otimes \mathbf{w}_p), \end{array} \right. \quad (14)$$

where \mathbf{w}_p is the weight vector of assets in p , \mathbf{E} is the $(N \times 1)$ vector of expected returns, $\mathbf{\Omega}$ is the $(N \times N)$ matrix of covariance, $\mathbf{\Sigma}$ is the $(N \times N^2)$ global matrix of coskewness, and $\mathbf{\Gamma}$ is the $(N \times N^3)$ global matrix of *cokurtosis* between all risky security returns, and

the sign \otimes standing for the symbol of the Kronecker product¹⁰.

In this representation, matrices Σ and Γ are built using the following scheme:

$$\begin{cases} \Sigma_{(N \times N^2)} = (\Sigma_1 \ \Sigma_2 \ \cdots \ \Sigma_N) \\ \Gamma_{(N \times N^3)} = (\Gamma_{11} \ \Gamma_{12} \ \cdots \ \Gamma_{1N} \mid \Gamma_{21} \ \Gamma_{22} \ \cdots \ \Gamma_{2N} \mid \dots \mid \Gamma_{N1} \ \Gamma_{N2} \ \cdots \ \Gamma_{NN}) \end{cases} \quad (15)$$

where Σ_k and Γ_{kl} are the $(N \times N)$ associated sub-matrices of Σ and Γ , with single elements $(s_{ijk})_{(i,j)=[1,\dots,N]^2}$ and $(\kappa_{ijkl})_{(i,j)=[1,\dots,N]^2}$, for any given coupled $(k \times l) = (IN^*)^2$.

We now propose herein a strictly equivalent notation for defining C-moments. Let us first start by defining the n -th recursive convolution matrix operator of a function $H(\cdot)$, denoted *per* convention $H^{(\odot n)}(\mathbf{w}_p)$, for $n \in IN^*$, such as:

$$H^{(\odot n)}(\cdot) \equiv \underbrace{H \{H [\dots H (\cdot)]\}}_{n \text{ operations}}, \quad (16)$$

and with for $n = 0$, *per* definition, $H^{(\odot 0)}(\cdot) = Id(\cdot)$ the identity function.

Secondly, we can define the following recurrent relation, applied to weight \mathbf{w}_p , with $n \in IN$ (and with previous notations):

$$H^{(\odot n)}(\mathbf{w}_p) = \mathbf{Vec} \left[H^{[\odot(n-1)]}(\mathbf{w}_p) \times \mathbf{w}_p' \right] = H^{[\odot(n-1)]}(\mathbf{w}_p) \otimes \mathbf{w}_p, \quad (17)$$

¹⁰If \mathbf{A} is a $(M \times P)$ matrix and \mathbf{B} a $(N \times Q)$ matrix, the $(MN \times PQ)$ matrix $(\mathbf{A} \otimes \mathbf{B})$ is called the Kronecker product of \mathbf{A} and \mathbf{B} , and is defined as such:

$$\mathbf{A} \otimes \mathbf{B}_{(MN \times PQ)} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1P}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2P}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1}\mathbf{B} & a_{M2}\mathbf{B} & \cdots & a_{MP}\mathbf{B} \end{pmatrix},$$

where:

$$a_{np}\mathbf{B}_{(N \times Q)} = \begin{pmatrix} a_{np}b_{11} & a_{np}b_{12} & \cdots & a_{np}b_{1Q} \\ a_{np}b_{21} & a_{np}b_{22} & \cdots & a_{np}b_{2Q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{np}b_{N1} & a_{np}b_{N2} & \cdots & a_{np}b_{NQ} \end{pmatrix},$$

with a_{mp} and b_{nq} are the elements of matrices A and B , and $(m, n, p, q) = [1, \dots, M] \times [1, \dots, N] \times [1, \dots, P] \times [1, \dots, Q] \subset IN^{*4}$.

where the function $H(\cdot)$ is defined such as $H(\mathbf{W}_p) = \mathbf{Vec}(\mathbf{W}_p \times \mathbf{w}_p')$, with \mathbf{W}_p being a vector of transformed weights \mathbf{w}_p and $\mathbf{Vec}(\cdot)$ the operator that reshapes a $(N \times M)$ matrix in a $(NM \times 1)$ vector, with $(N, M) = IN^*2$.

Thirdly, define the (repeated) Hadamard product of returns on any set of n assets of the portfolio p under studies, such that (*per* convention) we have, with $n \in [2, \dots, 4]$:

$$\bigodot_{q=1}^n \check{\mathbf{R}}(a_{[q]}) \equiv \underbrace{\check{\mathbf{R}}(a_{[1]}) \odot \dots \odot \check{\mathbf{R}}(a_{[n]})}_{n \text{ terms}}, \quad (18)$$

$(T \times 1)$

where the sign \odot stands for the symbol of the (simple) Hadamard product¹¹, and with $\check{\mathbf{R}}(q)$ the q -th column of $\check{\mathbf{R}}$, $\check{\mathbf{R}} = \mathbf{R} - (\mathbf{E} \times \mathbf{1}_T')'$ being the $(T \times N)$ matrix of centered returns, \mathbf{R} the $(T \times N)$ matrix of returns on the N assets, $\mathbf{1}_T$ the $(T \times 1)$ unit vector and the $a_{[q]}$ (with $q \in [1, \dots, n] \subset [1, \dots, N]$) being the ranks (column number) of the assets in the matrix of excess returns $\check{\mathbf{R}}$ (taken in any order), that we want to compute the related higher-order comoment, and which identify the location of a specific element in global matrices of higher-order comoments of individual stock returns.

With the two previous definitions and the recurrent relation, we are now able to define any (scalar) C-moment¹² of order n , denoted $m^n(R_p)$, as well as any related global (higher-

¹¹The $(N \times M)$ Hadamard product matrix $(\mathbf{A} \odot \mathbf{B})$ of two similar $(N \times M)$ matrices \mathbf{A} and \mathbf{B} , is defined as such:

$$\begin{aligned} (\mathbf{A} \odot \mathbf{B})_{(n,m)} &= \mathbf{A}_{(n,m)} \times \mathbf{B}_{(n,m)} \\ &\quad (1 \times 1) \quad \Longleftrightarrow \\ \mathbf{A} \odot \mathbf{B} &= \begin{pmatrix} a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1M}b_{1M} \\ a_{21}b_{21} & a_{22}b_{22} & \cdots & a_{2M}b_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1}b_{N1} & a_{N2}b_{N2} & \cdots & a_{NM}b_{NM} \end{pmatrix}, \end{aligned}$$

$(N \times M)$

where a_{nm} and b_{nm} are elements of A and B , with $(n, m) = [1, \dots, N] \times [1, \dots, M] \subset IN^*2$.

¹²This notation can furthermore be extended to higher higher-order L-comoments with no difficulty. For instance, the fifth-order Linear moment and comoment read (with the same notations):

$$\begin{cases} m^5(R_p) = \mathbf{w}_p' \times \mathbf{M}^5 \times H^{(\odot 5)}(\mathbf{w}_p) \\ m_{ijklm} = \mathbf{M}^5_{[i, (j-1)N^3 + (k-1)N^2 + (l-1)N + m]} = T^{-1} \times \mathbf{1}_T' \times [\check{\mathbf{R}}(i) \odot \check{\mathbf{R}}(j) \odot \check{\mathbf{R}}(k) \odot \check{\mathbf{R}}(l) \odot \check{\mathbf{R}}(m)]. \end{cases}$$

Since L-moments always exist, this expression may interestingly lead to further future refinements.

order) comoment $(N \times N^{n-1})$ matrix \mathbf{M}^n , with elements¹³ $\mathbf{M}_{(i,j)=(N \times N^{n-1})}^n$, with $j = \sum_{q=1}^{n-1} (a_{[q]} N^{n-1-q})$, being such that, with $n \in [2, \dots, 4]$ (and with previous notations):

$$\begin{cases} m_{(1 \times 1)}^n(R_p) &= \mathbf{w}_p' \times \mathbf{M}^n \times H^{[\odot(n-2)]}(\mathbf{w}_p) \\ m_{(1 \times 1)}^{a_{[1]} \dots a_{[n]}} &= \mathbf{M}^n \left[a_{[1]}, \sum_{q=1}^{n-1} (a_{[q+1]} - 1) \times N^{n-1-q} \right] = T^{-1} \times \mathbf{1}_T' \times \left[\bigodot_{q=1}^n \check{\mathbf{R}}(a_{[q]}) \right]. \end{cases} \quad (19)$$

Using this generic writing, the system of C-moments in equation (14) then can simply be expressed as such (with previous notations):

$$\begin{cases} E(R_p) = \mathbf{w}_p' \mathbf{E} \times 1 \\ \sigma^2(R_p) = \mathbf{w}_p' \times \mathbf{\Omega}_{\mathbf{w}_p} = \mathbf{w}_p' \times (\mathbf{\Omega} \times \mathbf{w}_p) \equiv \mathbf{w}_p' \times [\mathbf{\Omega} \times H^{(\odot 0)}(\mathbf{w}_p)] \\ m^3(R_p) = \mathbf{w}_p' \times \mathbf{\Sigma}_{\mathbf{w}_p} = \mathbf{w}_p' \times [\mathbf{\Sigma} \times [H(\mathbf{w}_p)]] \equiv \mathbf{w}_p' \times [\mathbf{\Sigma} \times H^{(\odot 1)}(\mathbf{w}_p)] \\ \kappa^4(R_p) = \mathbf{w}_p' \times \mathbf{\Gamma}_{\mathbf{w}_p} = \mathbf{w}_p' \times \{\mathbf{\Gamma} \times H[H(\mathbf{w}_p)]\} \equiv \mathbf{w}_p' \times [\mathbf{\Gamma} \times H^{(\odot 2)}(\mathbf{w}_p)], \end{cases} \quad (20)$$

where $\mathbf{\Sigma}$ and $\mathbf{\Gamma}$ are (still) the global $(N \times N^2)$ coskewness and $(N \times N^3)$ *cokurtosis* matrices (strictly equivalent to their previous tensor forms), but with each element being expressed in the new notation such as, $\forall(i, j, k, l) = [1, \dots, N]^4$ (with previous notations):

$$\begin{cases} \sigma_{ij} &= \mathbf{\Omega}_{[i,j]} = T^{-1} \times \mathbf{1}_T' \times [\check{\mathbf{R}}(i) \odot \check{\mathbf{R}}(j)] \\ m_{ijk} &= \mathbf{\Sigma}_{[i,(j-1)N+k]} = T^{-1} \times \mathbf{1}_T' \times [\check{\mathbf{R}}(i) \odot \check{\mathbf{R}}(j) \odot \check{\mathbf{R}}(k)] \\ \kappa_{ijkl} &= \mathbf{\Gamma}_{[i,(j-1)N^2+(k-1)N+l]} = T^{-1} \times \mathbf{1}_T' \times [\check{\mathbf{R}}(i) \odot \check{\mathbf{R}}(j) \odot \check{\mathbf{R}}(k) \odot \check{\mathbf{R}}(l)]. \end{cases}$$

The previous new compact notation exhibits some advantages compared to the traditional one (Athayde and Flôres (2002)), which essentially relied on a tensor notation of

¹³As pointed out by Jondeau, Poon and Rockinger (2007), many elements are the same in the matrices $\mathbf{\Sigma}$ and $\mathbf{\Gamma}$: only $N(N+1)(N+2)/6$ out of N^3 for $\mathbf{\Sigma}$ and $N(N+1)(N+2)(N+3)/24$ out of N^4 for $\mathbf{\Gamma}$ are different. Imposing the $a_{[q]}$, $q \in [1, \dots, n] \subset [1, \dots, N]$, the ranks of the assets (that we want to compute a specific higher-order comoment) to be ordered as in the matrix $\check{\mathbf{R}}$ (and not free as in the above notation), allows us to only provide the distinct elements of the matrix \mathbf{M}^n . For illustration purpose, we note that for the 162 stocks used in the following empirical application (see below), the coskewness and *cokurtosis* matrices contain more than 3 millions and 650 millions redundant terms. Not computing these terms and weighting the distinct ones according to the number of their repetitions, leads us to divide by ten or so the computation time of matrix $\mathbf{\Sigma}$ and $\mathbf{\Gamma}$. Moreover, the parsimonious new approach permits us to handle large-scale portfolio problems more easily.

coskewness and *cokurtosis* of asset returns on a portfolio, its global skewness and *kurtosis*. This new notation, more “computational-oriented”, is strictly equivalent to the previous one (since the Hadamard product terms are all included in the Kronecker matrices of weights), but first can be generalized in a more compact form in a recursive manner from the first moment to the n -th higher than the fourth, and, furthermore, gives a direct expression of all elements of the skewness and *kurtosis* matrices; secondly, it still allows us to disentangle the weight and the asset return impacts on higher-order moments; thirdly, it explicits the links between higher-order comoments and fourthly, it uses only traditional simple (low-level) pre-programmed operators (for building matrices of coskewness and *cokurtosis*) and thus appears to lead to a substantial overall gain in terms of execution time¹⁴.

The traditional multi-moment asset allocation setting now revisited, we shall adapt it hereafter to the analogues of the above C-moments of returns on any portfolio p , in the robust framework of Linear moments.

B. Higher-order L-comoments of Portfolio Returns

Let us also recall that \mathbf{R} , \mathbf{E} and $\check{\mathbf{R}}$ denote respectively the $(T \times N)$, $(N \times 1)$ and $(T \times N)$ vectors of effective returns, expected returns and centered realized returns on the N risky assets. In the context of robust L-comoment computations, the expectation (denoted $E(R_{\mathbf{w}_p})$, with $R_{\mathbf{w}_p}$ being a random variable) of the $(N \times 1)$ vector of observed returns, $\mathbf{R}_{\mathbf{w}_p}$, on the portfolio defined by its weight \mathbf{w}_p , the matrices $\mathbf{\Omega}_{\mathbf{w}_p}^{(L)}$, $\mathbf{\Sigma}_{\mathbf{w}_p}^{(L)}$ and $\mathbf{\Gamma}_{\mathbf{w}_p}^{(L)}$, representing respectively the $(N \times 1)$ vectors of the L-covariance, L-coskewness and L-*cokurtosis*¹⁵

¹⁴When we empirically double-checked the strict equivalence of the two alternative notations of higher-order comoments (*i.e.* the traditional Kronecker *versus* the new recursive Hadamard forms), it appears that the new one leads to a (limited) reduction of the execution time (by 7% or so), representing, however, several hours of spared computation time in large-scale portfolio selection applications. Moreover, both previous notations using C-comoments include a lot of redundant information in global matrices of coskewness and *cokurtosis* (see previous Footnote). Only computing the distinct elements may represent a 90% economy of execution time in a large-scale problem (see below).

¹⁵Note that the dimension of the second C-comoment matrix $\mathbf{\Omega}$ and those of the second L-comoment matrix $\mathbf{\Omega}_{\mathbf{w}_p}^{(L)}$, are different; the same is true for higher-order comoments (*i.e.* $\mathbf{\Sigma}_{\mathbf{w}_p}$ and $\mathbf{\Sigma}_{\mathbf{w}_p}^{(L)}$, $\mathbf{\Gamma}_{\mathbf{w}_p}$ and $\mathbf{\Gamma}_{\mathbf{w}_p}^{(L)}$).

of the security returns with the returns on the portfolio p , can be defined¹⁶ as (with previous notations):

$$\left\{ \begin{array}{ll} E(R_{\mathbf{w}_p}) &= \mathbf{w}_p' \mathbf{E} \\ \Omega_{\mathbf{w}_p}^{(L)} &= 2 E \{ \tilde{\mathbf{R}} \times \{ F(\mathbf{R}_{\mathbf{w}_p}) - E[F(\mathbf{R}_{\mathbf{w}_p})] \} \} \\ \Sigma_{\mathbf{w}_p}^{(L)} &= 6 E \{ \tilde{\mathbf{R}} \times \{ F(\mathbf{R}_{\mathbf{w}_p}) - E[F(\mathbf{R}_{\mathbf{w}_p})] \}^2 \} \\ \Gamma_{\mathbf{w}_p}^{(L)} &= 20 E \{ \tilde{\mathbf{R}} \times \{ F(\mathbf{R}_{\mathbf{w}_p}) - E[F(\mathbf{R}_{\mathbf{w}_p})] \}^3 \}, \end{array} \right. \quad (21)$$

where $F(\cdot)$ is the distribution of the random variable $R_{\mathbf{w}_p}$.

Using the covariance representation of L-moments defined in Equation (6) and the bilinear property of the covariance operator (see, for instance, Yitzhaki (2003)), the various population L-(co)moments of the returns on any attainable portfolio are respectively given by (with previous notations):

$$\left\{ \begin{array}{l} \lambda_1(R_{\mathbf{w}_p}) = E(R_{\mathbf{w}_p}) = \sum_{i=1}^N w_i E(R_i) \\ \lambda_2(R_{\mathbf{w}_p}) = 2Cov[R_{\mathbf{w}_p}, F(R_{\mathbf{w}_p})] = \sum_{i=1}^N w_i \lambda\text{-Cov}(R_i, R_{\mathbf{w}_p}) \\ \lambda_3(R_{\mathbf{w}_p}) = 6Cov\{R_{\mathbf{w}_p}, \{F(R_{\mathbf{w}_p}) - E[F(R_{\mathbf{w}_p})]\}^2\} = \sum_{i=1}^N w_i \lambda\text{-Cos}(R_i, R_{\mathbf{w}_p}) \\ \lambda_4(R_{\mathbf{w}_p}) = Cov\left\{R_{\mathbf{w}_p}, 20\{F(R_{\mathbf{w}_p}) - E[F(R_{\mathbf{w}_p})]\}^3 - 3\{F(R_{\mathbf{w}_p}) - E[F(R_{\mathbf{w}_p})]\}\right\} \\ \quad = \sum_{i=1}^N w_i [\lambda\text{-Cokurt}(R_i, R_{\mathbf{w}_p}) - 3(2)^{-1} \lambda\text{-Cov}(R_i, R_{\mathbf{w}_p})], \end{array} \right. \quad (22)$$

¹⁶The first L-moment strictly corresponds to the arithmetic mean return. Since we propose robust statistics to assess portfolio return peculiarities, it was natural to wonder if the use of the alternative measures of expected performance could have been more appropriate in a portfolio choice context. Some authors have shown that a bias could arise when using an arithmetic mean; a geometric mean is certainly more accurate when estimating a long term expected return. But with the one-week horizon used in our application, this bias is not relevant (see Hughson, Stutzer and Yung (2006)). Other authors prefer to use a robust statistic for a location parameter (such as the median) instead of the classical mean when estimating expected returns (see McCulloch (2003)). Two reasons motivate us to stay here in the classical paradigm. First, the median (or the first Trimmed L-moment) neglects the impact of (some of) the extreme returns on the performance; considering the median could thus in some cases blur the investor perception. Secondly, we also computed Four-moment optimal portfolios using the median, but we did not find any clear difference between the two approaches (same overall conclusions in Section 4 apply; see also Footnote 21).

and, for any asset i , with $i \in [1, \dots, N]$:

$$\begin{cases} \lambda\text{-Cov}(R_i, R_{\mathbf{w}_p}) = 2 E \{ [R_i - E(R_i)] \times \{F(R_{\mathbf{w}_p}) - E[F(R_{\mathbf{w}_p})]\} \} \\ \lambda\text{-Cos}(R_i, R_{\mathbf{w}_p}) = 6 E \{ [R_i - E(R_i)] \times \{F(R_{\mathbf{w}_p}) - E[F(R_{\mathbf{w}_p})]\}^2 \} \\ \lambda\text{-Cokurt}(R_i, R_{\mathbf{w}_p}) = 20 E \{ [R_i - E(R_i)] \times \{F(R_{\mathbf{w}_p}) - E[F(R_{\mathbf{w}_p})]\}^3 \} \\ E[F(R_{\mathbf{w}_p})] = 1/2, \end{cases}$$

where $\lambda\text{-Cov}(\cdot)$, $\lambda\text{-Cos}(\cdot)$ and $\lambda\text{-Cokurt}(\cdot)$ correspond respectively to the L-covariance, L-coskewness and L-cokurtosis between any asset i return and the portfolio p return defined by its holdings \mathbf{w}_p .

The L-covariance, L-coskewness and L-cokurtosis of portfolio returns used in Equation (21) can be written in a compact matrix format¹⁷ as such, for $n \geq 2$ (with previous notations):

$$\mathbf{M}_{\mathbf{w}_p}^{(L) n} = p_{n-1, n-1}^* T^{-1} \left\{ \check{\mathbf{R}} \odot \left\{ \bigodot_{q=1}^{n-1} \left\{ q^0 \times \left[F(\mathbf{R}_{\mathbf{w}_p}) \times \mathbf{1}'_N - 1/2 \right] \right\} \right\} \right\}' \times \mathbf{1}_T, \quad (23)$$

with $\mathbf{M}_{\mathbf{w}_p}^{(L) 2} = \mathbf{\Omega}_{\mathbf{w}_p}^{(L)}$, $\mathbf{M}_{\mathbf{w}_p}^{(L) 3} = \mathbf{\Sigma}_{\mathbf{w}_p}^{(L)}$, $\mathbf{M}_{\mathbf{w}_p}^{(L) 4} = \mathbf{\Gamma}_{\mathbf{w}_p}^{(L)}$ and $p_{n-1, n-1}^* = [2(n-1)]! [(n-1)!]^{-2}$ a factor being equal to the $(n-1)$ -th (highest) coefficient of the shifted orthogonal Legendre polynomial $P_{n-1}^*(\cdot)$ of degree $n-1$, as previously defined in equation (1).

The L-moments, written in a generic manner such as (with previous notations):

$$\begin{cases} \lambda_1(R_{\mathbf{w}_p}) = \mathbf{w}_p' \mathbf{E} \\ \lambda_2(R_{\mathbf{w}_p}) = \mathbf{w}_p' \mathbf{\Omega}_{\mathbf{w}_p}^{(L)} \\ \lambda_3(R_{\mathbf{w}_p}) = \mathbf{w}_p' \mathbf{\Sigma}_{\mathbf{w}_p}^{(L)} \\ \lambda_4(R_{\mathbf{w}_p}) = \mathbf{w}_p' \mathbf{\Gamma}_{\mathbf{w}_p}^{(L)} - 3(2)^{-1} \mathbf{w}_p' \mathbf{\Omega}_{\mathbf{w}_p}^{(L)}, \end{cases} \quad (24)$$

These L-moments can be reformulated in a (even) more compact manner¹⁸ reading, for $n \geq 2$:

$$\lambda_n(R_{\mathbf{w}_p}) = \mathbf{w}_p' \mathbf{M} \mathbf{c}_{\mathbf{w}_p}^{(L) n}, \quad (25)$$

¹⁷Compared to higher-order C-comoment writings, we note here that L-comoment analogues do not contain any redundant element.

¹⁸The use of higher-order L-comoments, computed with this compact writing instead of the previous related C-moment one, leads empirically to a drastic reduction in the execution time (divided by about four or so) in our general Goal Attainment problem (see below), whilst computing the first L-moments instead of the first C-moments of portfolio returns is approximately 40% faster.

with:

$$\mathbf{Mc}_{\mathbf{w}_p}^{(L) \ 2} = T^{-1} \left\{ \check{\mathbf{R}} \odot \left\{ \left\{ P_{n-1}^* [F(\mathbf{R}_{\mathbf{w}_p})] - E \{ P_{n-1}^* [F(\mathbf{R}_{\mathbf{w}_p})] \} \right\} \times \mathbf{1}'_N \right\} \right\}' \times \mathbf{1}_T,$$

where $\mathbf{Mc}_{\mathbf{w}_p}^{(L) \ 2} = \mathbf{\Omega}_{\mathbf{w}_p}^{(L)}$, $\mathbf{Mc}_{\mathbf{w}_p}^{(L) \ 3} = \mathbf{\Sigma}_{\mathbf{w}_p}^{(L)}$, and $\mathbf{Mc}_{\mathbf{w}_p}^{(L) \ 4} = \mathbf{\Gamma}_{\mathbf{w}_p}^{(L)} - 3(2)^{-1} \mathbf{\Omega}_{\mathbf{w}_p}^{(L)}$.

Since we now have the complete characterization of all L-moments of portfolio returns, we can then define hereafter more precisely the set of efficient portfolios.

C. Higher-order L-moments and the Efficient Frontier Definition

We now consider the problem of an investor selecting a portfolio from N risky assets (with $N \geq 4$) in the four L-moment framework. We assume that the investor does not have access to a riskless asset, and that the portfolio weights sum to one. In addition, we impose¹⁹ a no short-sale portfolio constraint.

Any portfolio p is here entirely defined by $\mathbf{w}_p \in IR^N$, the vector of weights of assets, and the set of the attainable portfolios \mathfrak{A} can then be expressed as follows:

$$\mathfrak{A} = \left\{ \mathbf{w} \in IR^N : \mathbf{w}' \mathbf{1} = 1 \text{ and } \mathbf{w} \geq \mathbf{0} \right\}, \quad (26)$$

where \mathbf{w}' is the $(1 \times N)$ transposed vector of the investor's holdings in the various risky assets, $\mathbf{1}$ is the $(N \times 1)$ unitary vector and $\mathbf{0}$ is the $(N \times 1)$ null vector

As in Markowitz (1952), the definition of moments of portfolio's returns indeed leads to the disposal representation of the set of the feasible portfolios, denoted \mathfrak{F} , in the extended four L-moment space²⁰ (see Briec, Kerstens and Lesourd (2004), and Briec, Kerstens and Jokung (2007)), reading:

$$\mathfrak{F} = \{\boldsymbol{\lambda}_{\mathbf{w}} : \mathbf{w} \in \mathfrak{A}\} + [(-IR_+) \times IR_+ \times (-IR_+) \times IR_+], \quad (27)$$

¹⁹This assumption can however be relaxed to some extent with no loss of generality (see Footnote 28).

²⁰For the sake of simplicity, for each point of the efficient frontier, we choose to consider “portfolios” even if we should only speak about “classes of equivalence induced by these portfolios” (see end of Appendix 2).

where $\lambda_{\mathbf{w}}$ is the (4×1) vector of the first four L-moments²¹ of the portfolio return $R_{\mathbf{w}}$, *i.e.*:

$$\lambda_{\mathbf{w}} = [\lambda_1(R_{\mathbf{w}}); \lambda_2(R_{\mathbf{w}}); \lambda_3(R_{\mathbf{w}}); \lambda_4(R_{\mathbf{w}})]'.$$

This disposal representation is necessary here to ensure the convexity of the feasible portfolio set in the four L-moment space (see Briec and Kerstens, (2007), and Zhang (2008) for an illustration on consequences of the non-convexity of the set of portfolios on the choice of optimal ones).

We define a strict (generic) order relation, denoted by \succ , on IR^4 , that is for any $(\lambda, \tilde{\lambda}) \in (IR^4)^2$:

$$\lambda \succeq \tilde{\lambda} \iff [\lambda_1 \geq \tilde{\lambda}_1, \lambda_2 \leq \tilde{\lambda}_2, \lambda_3 \geq \tilde{\lambda}_3, \lambda_4 \leq \tilde{\lambda}_4], \quad (28)$$

altogether with a strict relation, denoted by \succ , as such:

$$\lambda \succ \tilde{\lambda} \iff [\lambda_1 > \tilde{\lambda}_1, \lambda_2 < \tilde{\lambda}_2, \lambda_3 > \tilde{\lambda}_3, \lambda_4 < \tilde{\lambda}_4]. \quad (29)$$

The four L-moment weakly efficient frontier \mathfrak{L} is then defined as follows:

$$\mathfrak{L} = \left\{ \lambda_{\mathbf{w}} \in \mathfrak{F} : \forall \tilde{\lambda} \in IR^4, \tilde{\lambda} \succ \lambda_{\mathbf{w}} \Rightarrow \tilde{\lambda} \notin \mathfrak{F} \right\}. \quad (30)$$

whilst the four L-moment strong efficient frontier \mathfrak{M} is defined as follows:

²¹As mentioned earlier, we make use here of the traditional L-moments instead of other variants (such as the Trimmed L-moments for instance). Despite the fair argument by Darolles, Gouriéroux and Jasiak (2008), highlighting that for large samples the “...Trimmed L-moments of order 1 bridge the mean and the median”, we choose not to use them in a portfolio choice context for three main reasons. As a first reason, it is clear that Trimmed or Quantile L-moments provide more accurate estimates of the underlying distribution characteristics when focusing on extremes in a risk estimation exercise for instance; it is far from obvious, however, that the influence of large deviations should be too much reduced in a portfolio choice framework, since extremes should have - in a sense - some influence on the first and the second moments of returns. Deleting some really “bad” returns of a hedge fund record for instance and computing the related first Trimmed L-moment will probably result in an upward bias of the future anticipated performance of the fund. Secondly, if there exist more robust alternatives to the sensitive mean operator for the location parameter of a distribution (such as the median or the first Trimmed L-moment of order n), a more realistic and safer approach in a portfolio choice framework would probably be to stick with the first conventional moment, since it is more in line with the value of the portfolio, as directly perceived by the investor and recommended by market authorities. The third and last reason is that replacing mean by median in our preliminary tests, did not lead to huge differences in our application; that is to say that the main general conclusions of the following empirical study (see Section 4) stay the same in our long-only plain vanilla stock application (see also Footnote 16).

$$\mathfrak{M} = \left\{ \lambda_{\mathbf{w}} \in \mathfrak{D} : \forall \tilde{\lambda} \in IR^4, [(\tilde{\lambda} \succeq \lambda_{\mathbf{w}}) \text{ and } (\tilde{\lambda} \neq \lambda_{\mathbf{w}})] \Rightarrow \tilde{\lambda} \notin \mathfrak{D} \right\}. \quad (31)$$

The strong efficient portfolio frontier is then the set of portfolios, defined by their weights \mathbf{w} , such that the associated L-moment quadruplet is not strictly dominated in the four-dimensional space. It is then given in the simplex by:

$$\mathfrak{E} = \{\mathbf{w} \in \mathfrak{A} : \lambda_{\mathbf{w}} \in \mathfrak{M}\}. \quad (32)$$

By analogy with tools developed in the field of the production theory (see Luenberger (1995)), the next section introduces the so-called shortage function as an indicator of a portfolio L-moment (in)efficiency, and presents the non-convex higher-order L-moment version of the portfolio optimization program. The solution of the resulting program will be called the four L-moment Efficient Set.

D. The Shortage Function and the Robust Efficient Frontier

In order to obtain the set of portfolios of the weakly efficient frontier, we need to resolve a multi-objective optimization problem. That is maximize simultaneously the first and the third order L-moments and minimize the second and the fourth L-moments. Several methods allowing the solution of multi-objective problems have been proposed in the literature. Goal Programming, a branch of multi-objective optimization theory introduced by Charnes, Cooper and Ferguson (1955), operates with a set of linear objective functions. Since higher L-moments are clearly non-linear, such an approach should be banned in our case. Another intuitive approach is to aggregate all objectives in a global weighted target function. Optimal portfolios could then be obtained using a traditional non-linear optimizer, but one still needs to specify the importance of the different objectives, which is finally equivalent to introducing a utility function into the problem.

In the following, we choose to use a sequential quadratic programming method to solve our problem. The introduction of a shortage function enables us to optimize simultaneously all the objectives, since this latter function measures the distance between some points of the possibility set and the efficient frontier (see Luenberger (1995)). The properties of the set of the portfolio return moments on which the shortage function is defined have already been discussed in the mean-variance plane by Briec, Kerstens and Lesourd (2004) and in the higher moment space by Jurczenko, Maillet and Merlin (2006), Ryoo (2007), Briec and Kerstens (2007), Briec, Kerstens and Jokung (2007), and Yu, Wang and Lai (2008). Their definitions can be extended to obtain a portfolio efficiency indicator in

the four L-moment framework. The shortage function associated to any portfolio \mathbf{w} in the feasible set \mathfrak{A} , with reference to the direction vector $\mathbf{g} = (g_1; g_2; g_3; g_4)$, with $\mathbf{g} \in (IR_+ \times IR_- \times IR_+ \times IR_-) \setminus \{0\}$, in the mean-L-variance-L-skewness-L-kurtosis space, is the real-valued function $S_{\mathbf{g}}(.)$ defined such as:

$$S_{\mathbf{g}}(\mathbf{w}) = \sup_{\delta \in IR_+} \{\delta : (\boldsymbol{\lambda}_{\mathbf{w}} + \delta \mathbf{g}) \in \mathfrak{F}\}. \quad (33)$$

We have the following existence result regarding the shortage function.

Proposition 1. *For every $\mathbf{g} \in (IR_+ \times IR_- \times IR_+ \times IR_-) \setminus \{0\}$ and every $\mathbf{w} \in \mathfrak{A}$, there exists a unique element δ^* , $\delta^* \in IR_+$, such that:*

$$S_{\mathbf{g}}(\mathbf{w}) = \boldsymbol{\lambda}_{\mathbf{w}} + \delta^* \mathbf{g}. \quad (34)$$

See Appendix 2 for proof.

The use of the shortage function in the mean-L-variance-L-skewness-L-kurtosis can unfortunately only guarantee the weak efficiency for a portfolio since it does not exclude projections on the vertical and horizontal parts of the frontier allowing portfolios for additional improvements, but hopefully constraints can be easily imposed in the practical implementation for searching only strong efficient portfolios²² (see below).

The disposal representation of the feasible portfolio set can now be used for deriving the lower bound of the true but unknown four L-moment efficient frontier, through the computation of the associated portfolio shortage function. Let us consider a specific portfolio p , defined by its vector of weights denoted \mathbf{w}_p , compound from a set of N assets and whose performance needs to be evaluated in the four L-moment dimensions. We then define the function $\Phi_{\mathbf{w}_p, \mathbf{g}}(.)$ from \mathfrak{A} to IR_+ by:

$$\Phi_{\mathbf{w}_p, \mathbf{g}}(\mathbf{w}) = \sup_{\delta \in IR_+} \{\delta : \boldsymbol{\lambda}_{\mathbf{w}} \succeq (\boldsymbol{\lambda}_{\mathbf{w}_p} + \delta \mathbf{g})\}. \quad (35)$$

We also remark that:

$$\Phi_{\mathbf{w}_p, \mathbf{g}}(\mathbf{w}) = \min_{i \in [1, \dots, 4]} \{[\lambda_i(R_{\mathbf{w}}) - \lambda_i(R_{\mathbf{w}_p})] (g_i)^{-1}\}, \quad (36)$$

where g_i is the i -th component, with $i = [1, \dots, 4]$, of the direction vector \mathbf{g} .

²²Wierzbicki (1986) proposes a theorem of characterization of strong efficient solutions (based on the evaluation of marginal substitution rates between objectives) that allows us to remove a part of weak efficient portfolios from the set of optimal solutions. However, we did not want at this stage to impose any explicit preference specification in a particular utility function setting.

The function $S_{\mathbf{g}}$ is related to the function $\Phi_{\mathbf{w}_p, \mathbf{g}}(.)$ by the following relation:

$$S_{\mathbf{g}}(\mathbf{w}_p) = \sup_{\mathbf{w} \in \mathfrak{A}} \{ \Phi_{\mathbf{w}_p, \mathbf{g}}(\mathbf{w}) \}. \quad (37)$$

Using the Goal Attainment method (see Gembicki and Haimes (1975) for a general comprehensive presentation), the shortage function for this portfolio is then computed by solving the following non-linear optimization program $\mathfrak{P}_{\mathbf{w}_p, \mathbf{g}}$:

$$\mathbf{w}^* = \underset{(\mathbf{w}, \delta) \in (\mathfrak{A} \times IR_+)}{\text{ArgMax}} \{ \Phi_{\mathbf{w}_p, \mathbf{g}}(\mathbf{w}) \}, \quad (38)$$

where \mathbf{w}^* is a $(N \times 1)$ weakly efficient portfolio weight vector that (weakly) maximizes the expected performance, L-variance, L-skewness, and L-kurtosis relative improvements over the evaluated portfolio p in the direction vector \mathbf{g} .

Using the vectorial notations of the portfolio return higher L-moments in (24) and using the first four L-moments of the specific evaluated portfolio \mathbf{w}_p in the expression of the direction vector \mathbf{g} , the non-parametric portfolio optimization program (38) can then be written in a restated version²³ such as (with previous notations):

$$\begin{aligned} \mathbf{w}^* = & \underset{(\mathbf{w}, \delta) \in (\mathfrak{A} \times IR_+)}{\text{ArgMax}} \{ \delta \} \\ s.t. & \begin{cases} \lambda_1(R_{\mathbf{w}_p}) + \delta \lambda_1(R_{\mathbf{w}_p}) \leq \mathbf{w}' \mathbf{E} \\ \lambda_2(R_{\mathbf{w}_p}) - \delta \lambda_2(R_{\mathbf{w}_p}) \geq \mathbf{w}' \boldsymbol{\Omega}_{\mathbf{w}}^{(L)} \\ \lambda_3(R_{\mathbf{w}_p}) + \delta \lambda_3(R_{\mathbf{w}_p}) \leq \mathbf{w}' \boldsymbol{\Sigma}_{\mathbf{w}}^{(L)} \\ \lambda_4(R_{\mathbf{w}_p}) - \delta \lambda_4(R_{\mathbf{w}_p}) \geq \mathbf{w}' \boldsymbol{\Gamma}_{\mathbf{w}}^{(L)} - 3(2)^{-1} \mathbf{w}' \boldsymbol{\Omega}_{\mathbf{w}}^{(L)}, \end{cases} \end{aligned} \quad (39)$$

with:

$$\mathbf{g} = [\lambda_1(R_{\mathbf{w}_p}); -\lambda_2(R_{\mathbf{w}_p}); \lambda_3(R_{\mathbf{w}_p}); -\lambda_4(R_{\mathbf{w}_p})]'$$

Due to the non-convex nature of the optimization program, we still need to establish the necessary and sufficient conditions showing that a local optimal solution of (38) is also a global *optimum*. We actually use the following result.

²³However, restricting (for instance) the search in the direction of an increase in the expected return may lead us to miss some peculiar portfolios that exhibit low (negative) expected returns but have other advantageous characteristics (namely low volatility, outstanding high skewness and rather small *kurtosis*). Nevertheless, in the specific context of portfolio selection, it is doubtful that such portfolios (lotteries) might be considered by any rational investor as optimal. We thus restrict our study to positive expected returns in the algorithm implementation that follows. The same problem arises when looking for a systematic increase in the skewness. In this case, we have to authorize the third coordinate of vector \mathbf{g} to be zero.

Proposition 2. If $(\mathbf{w}^*, \delta^*) \in (\mathfrak{A} \times IR_+)$ is a local solution of the following non-linear optimization program $\mathfrak{P}_{\mathbf{w}_p, \mathbf{g}}$:

$$\text{Max}_{(\mathbf{w}, \delta) \in (\mathfrak{A} \times IR_+)} \Phi_{\mathbf{w}_p, \mathbf{g}}(\mathbf{w}), \quad (40)$$

it is then a global solution.

See Appendix 2 for proof.

Indeed, despite the non-convex nature of the first four L-moment portfolio selection program, the shortage function maximization achieves a global optimum for the non-linear portfolio optimization program. This makes the shortage function technique superior to the other primal and dual approaches of the four moment efficient set, since the latter only guarantees to end with a local *optimum*. In the next section, we illustrate the shortage function technique in the case of a robust strong efficient portfolio selection.

For obtaining the set \mathfrak{M} , which corresponds to the set of portfolios whose first four L-moments are not simultaneously dominated, we then consider the evolutionary optimization problem $\mathfrak{P}_{\mathbf{w}_p, \mathbf{g}^j}$ at step j , $j \in IN^*$, such as (with previous notations):

$$\begin{aligned} \mathbf{w}^* &= \underset{(\mathbf{w}, \delta) \in (\mathfrak{A} \times IR_+)}{\text{ArgMax}} \{ \delta \} \\ \text{s.t.} &\begin{cases} \Delta_{1, \mathbf{w}_p}(\mathbf{w}, \delta, \mathbf{g}^j) \leq 0 \\ \Delta_{2, \mathbf{w}_p}(\mathbf{w}, \delta, \mathbf{g}^j) \geq 0 \\ \Delta_{3, \mathbf{w}_p}(\mathbf{w}, \delta, \mathbf{g}^j) \leq 0 \\ \Delta_{4, \mathbf{w}_p}(\mathbf{w}, \delta, \mathbf{g}^j) \geq 0, \end{cases} \end{aligned} \quad (41)$$

where $\Delta_{i, \mathbf{w}_p}(\cdot)$, with $i = [1, \dots, 4]$, being the following admissible portfolio directional differences:

$$\begin{cases} \Delta_{1, \mathbf{w}_p}(\mathbf{w}, \delta, \mathbf{g}^j) = \lambda_1 (R_{\mathbf{w}_p}) + \delta g_1^j - \mathbf{w}' \mathbf{E} \\ \Delta_{2, \mathbf{w}_p}(\mathbf{w}, \delta, \mathbf{g}^j) = \lambda_2 (R_{\mathbf{w}_p}) + \delta g_2^j - \mathbf{w}' \mathbf{\Omega}_{\mathbf{w}}^{(L)} \\ \Delta_{3, \mathbf{w}_p}(\mathbf{w}, \delta, \mathbf{g}^j) = \lambda_3 (R_{\mathbf{w}_p}) + \delta g_3^j - \mathbf{w}' \mathbf{\Sigma}_{\mathbf{w}}^{(L)} \\ \Delta_{4, \mathbf{w}_p}(\mathbf{w}, \delta, \mathbf{g}^j) = \lambda_4 (R_{\mathbf{w}_p}) + \delta g_4^j - \mathbf{w}' \mathbf{\Gamma}_{\mathbf{w}}^{(L)} + 3(2)^{-1} \mathbf{w}' \mathbf{\Omega}_{\mathbf{w}}^{(L)}, \end{cases}$$

with g_i^j is the i -th component of the direction vector \mathbf{g} at step j .

For illustration purposes, we start to solve the problem considering a portfolio p with a first simple direction function $\mathbf{g}^1 = (g_1^1, g_2^1, g_3^1, g_4^1)$, such as $\mathbf{g}^1 = (1, -1, 1, -1)$. Let (\mathbf{w}^1, δ^1) be a solution of $\mathfrak{P}_{\mathbf{w}_p, \mathbf{g}^1}$ and let $S^1 = \{i = [1, \dots, 4] : \Delta_{i, \mathbf{w}_p}(\mathbf{w}^1, \delta^1, \mathbf{g}^1) = 0\}$ be the set of indexes of saturated constraints. If not all constraints are saturated, *i.e.* $S^1 \neq \{1, \dots, 4\}$, then we consider the optimization problem $\mathfrak{P}_{\mathbf{w}_p, \mathbf{g}^2}$, where \mathbf{g}^2 is defined

by $g_i^2 = g_i^1$ if $i \notin S^1$, and $g_i^2 = 0$ if not. For a solution (\mathbf{w}^2, δ^2) of the problem $\mathfrak{P}_{\mathbf{w}^p, \mathbf{g}^2}$, if all constraints are saturated, then the portfolio defined by \mathbf{w}^2 is strongly efficient. Otherwise, we consider the new problem $\mathfrak{P}_{\mathbf{w}^p, \mathbf{g}^3}$ and we continue in the same manner until all constraints are saturated.

The idea of this optimization process is to saturate at each step at least one of the four constraints, whilst keeping saturated the already saturated constraints. In this aim, at each step, the starting point is the one obtained at the previous step, and the direction function is modified in order for the optimization process to follow a path along the weak efficient frontier. When all constraints are saturated, we then obtain a strongly efficient solution \mathbf{w}^* , *i.e.* a set of moment $\boldsymbol{\lambda}_{\mathbf{w}^*}$ that belongs to efficient portfolio set \mathfrak{M} .

We have here chosen for the starting direction function $\mathbf{g}^1 = (1, -1, 1, -1)$ for the sake of simplicity. Indeed, with such a fixed vector, the whole strong efficient frontier will be obtained by considering different starting points. However, for completing (and boosting) the optimization process, different starting points (portfolios p) and different first direction functions \mathbf{g}^1 in $(0; a] \times [-b; 0] \times (0; c] \times [-d; 0]$ are considered in the following practical implementation. The values of a , b , c and d are set in realistic ranges of potential improvements²⁴ and take into account the differences of scale between the four L-moments.

III. Data and Empirical Results

In the following empirical application, we explore a dataset of quotes of some of the most liquid European stocks, provided by *Bloomberg*, in the period from June 2001 to June 2006. The database consists of weekly Euro denominated returns of a sample of 162 stocks included in the *DJ European Stoxx* index. First, the selected stocks were not chosen randomly as in some previous studies, but were selected for obtaining a cylindric sample. The 162 stocks considered (representing approximately more than 1.3 trillion of Euros in terms of free float market capitalization at the end of the sample²⁵) consist of all stocks present over the whole sample period that have not experienced a corporate action (such as a stock split for instance) during the sample period. Secondly, as in Jondeau and Rockinger (2006a), we have chosen a weekly sample frequency. Indeed, it is worth emphasizing here that the frequency of data is often claimed to affect both departures from normality and serial correlation patterns of returns and volatilities. In our case,

²⁴These parameters are fixed hereafter to the maximum values of the L-moments found on individual assets in the sample.

²⁵that corresponds to a quarter or so of the total capitalization of the European index.

the frequency of observations²⁶ has been chosen in order to be low enough for being well adapted to an asset allocation problem (in which reallocations cannot happen very often), but high enough for keeping in the sample the main peculiarities of the financial returns, such as the skewed-heteroskedasticity *phenomenon*, that generally goes with unconditional asymmetric and leptokurtic underlying return distributions. Thirdly, the period of study is also rich in events (end of the internet bubble crash, the 9/11/2001 event, the market correction of May 2006...) and is characterized by a bear market on the first part of the sample (2001-2003), followed by a strong bull market (2003-2006). If we note that market performance is very high on the total period (with an annualized return of 18% for the *DJ European Stoxx* index, whilst the typical annualized return on the American stock DJI is equal to 5% on the period 1900-2006), we also remark that we have various market conditions (rallies, bear markets, booms and crashes) in the sample period (which is similar to those studied on the American stock market by Maringer and Parpas (2007)); it allows us to think that the sample is not too specific for our general purpose.

Since our aim is to evaluate whether, in some instances, the widely-used mean-variance *criterion* may be inappropriate in selecting the optimal portfolio weights, we shall check, before all, the univariate non-Gaussianity of the sample stock return series, using main classical Normality tests, namely Jarque-Bera, Kolmogorov-Smirnov, Lilliefors and Anderson-Darling tests. The Jarque-Bera test is one of the most used portmanteau Goodness-of-Fit measures of departure from normality and is based on sample skewness and *kurtosis*. The statistic of the test has an asymptotic Chi-squared density; however, it has been proven to have limited power in a small sample, because empirical counterparts of conventional third and fourth moments approach Gaussianity only very slowly. The second test we performed is the Kolmogorov-Smirnov one, which is another main classical Gaussianity test; it is based on the observed largest difference between the data-driven cumulative Empirical Density Function and the sample estimate of the Normal reference distribution. Correcting for the bias of using sample estimates of characteristic parameters of the reference law leads to the third test considered, which was proposed by Lilliefors. This test is designed with the null *hypothesis* that data come from a normally distributed population, when the tested *hypothesis* does not specify which normal distribution (*i.e.* without specifying expected value and variance). One of the peculiarities of this test is to be not too sensitive to outliers and thus more sensible to the adequation of the central part of the

²⁶In Jurczenko, Maillet and Merlin (2006), a monthly frequency (on hedge funds) was used, and first preliminary tests made here with daily returns (on stocks) showed no difference in overall general results of this article.

distribution. Finally, we also used the Anderson-Darling test, which is known for being one of the most powerful statistics for detecting departures from normality (see Stephen (1974), d’Agostino and Stephens (1986)). This test is based upon the concept that when given a hypothesized underlying distribution, the data can be transformed into a Uniform distribution, and is then crucially linked to tails of the data density.

We thus began by testing the effective (non-)normality of our stock return sample, focusing on different aspects of Gaussianity: using an explicit test on both skewness and *kurtosis* (according to Jarque-Bera tests), testing the largest inadequations (based on Kolmogorov-Smirnov tests), emphasizing the differences in centers of distributions (with Liliefors tests), as well as discrepancies in the tails (defined through Anderson-Darling tests). Not surprisingly, the vast majority of the original stock return series cannot indeed be considered as Gaussian (see below).

We then start the efficient portfolio search algorithm by generating randomly one thousand arbitrary portfolios, and then optimize each one in their four moments using our distance function approach in 9,637 directions²⁷. We present hereafter the empirical four-moment efficient frontiers and their projections in the various planes. Due to the large number of stocks considered in the efficient portfolios, this optimization problem belongs to a large-scale asset allocation problem class (see Perold (1984)). The main consequence is that we observe some strong discontinuities in the empirical efficient frontiers. This last feature is also intensified by the addition of strong short-sale constraints²⁸.

In the following, we will first start by evaluating the trade-offs between each pair of moments; we secondly analyze the efficient portfolios and their global characteristics, paying special attention to the higher moments that are neglected in the traditional analysis. In all the following representations, we further restrict the efficient portfolio set in considering only those with a positive mean (in a portfolio selection context) and a reasonable second annualized volatility²⁹. After having presented some efficient portfolio frontiers, we thirdly evaluate the optimality of (primal) potential efficient portfolios when grouping portfolios together based on their similarities in terms of L-moments, then when valuing

²⁷It corresponds to the number of combinations of 11 possible intensities (from 0 to 1 with a .1 step for each L-moment).

²⁸These constraints are realistic in the case of pure equity portfolios, but can easily be relaxed, to some extent, to relative constraints to the exposed capital as in Pástor and Stambaugh (2000) and So and Tse (2001).

²⁹The maximum annualized volatility level considered here is 40%, corresponding to a L-moment₂ inferior to .22. This restriction allows us to avoid representing the distortions caused by the no-short sale constraints for portfolios with high levels of volatility.

them according to a specific utility function in the dual approach. Fourthly, we will make the higher moments of efficient portfolios more appealing, transforming the real third and fourth L-moments to intensify potential distortions and evaluate the impact on optimal asset allocations. Finally, we will see that almost similar portfolios in terms of first and second L-moments may exhibit some differences in higher moments that might be, for the investor, interesting to control.

The Figures 2 to 7 represent the frontiers of the projections of the four L-moment efficient portfolios in two specific L-moment dimensions. In Figure 2, the traditional two-moment frontier (see Markowitz (1952)) is represented in its robust version. We find exactly the same type of shape for this restricted frontier: the relation between the mean and the $L\text{-moment}_2$ is a non-linear increasing one. If investors want to target better mean returns, they have to accept a higher dispersion of returns. When we look at the other characteristics of these efficient portfolios, we see that some high skewness portfolios appear for mild medium means and reasonable second L-moments (see Figures 2 and 3). We also remark that the second and fourth L-moments seem to vary in the same direction. This is confirmed in Figure 4, in which we represent the lowest fourth L-moments of the four L-moment efficient portfolios for each level of the $L\text{-moment}_2$. Thus, the value of the $L\text{-moment}_2$ of efficient portfolios depends positively on the level of the first and fourth L-moments. Accordingly, when investors tend to reach higher expected returns, they have to face an increase of the $L\text{-moment}_2$ and the $L\text{-moment}_4$ (see also Figure 5). In other words, increases of $L\text{-moment}_4$ go with increases of $L\text{-moment}_2$ for efficient portfolios. This result is consistent with those by Maringer and Parpas (2008)³⁰, and seems natural and intuitive when considering the traditional interpretation of $L\text{-moment}_2$ in terms of dispersion of returns and of the $L\text{-moment}_4$ as a consequence of the presence of extreme returns (that, *ceteris paribus*, entails a higher dispersion measure). The two positive relations (between the first and the second, the second and the fourth L-moments of four-moment efficient portfolios) impose a clear positive link between the first and fourth L-moments, as seen in Figure 6: the investors are here compensated for bearing more extreme risks.

Regarding now the relations between $L\text{-moment}_3$ and others L-moments for efficient portfolios, it seems far less clear as already seen in Figure 2 and indicated in the literature on the impact of skewness on the expected utility (see Brockett and Kahane (1992), Brockett and Garven (1998), and Christodoulakis and Peel (2006)). Moreover, frontiers present

³⁰despite exhibiting an apparent inverse relation since they used the fourth standardized moment. By construction, standardizing the fourth moment necessarily leads to an inverse relation with volatility.

large discontinuities as already pointed out by Athayde and Flôres (2002)³¹. In Figure 6, the highest mean efficient portfolios for each level of L-moment₃ are presented. We first note that the maximum mean return portfolio has a negative skewness. That is: willing high mean returns has a price in terms of L-moment₃ (called a skewness *premium*; see Post, Vliet and Levy (2008) and relevant literature). From Figure 6 (and Figure 2), it also appears that the highest L-moment₃ efficient portfolios have medium mean returns (and L-moment₂). Finally, the lowest mean (L-moment₂) efficient portfolios are associated with a low (negative) asymmetry as shown in Figure 3 and Figure 6, the former representing the link between the highest L-moment₃ for various L-moment₂ of the four L-moment efficient portfolios. An interesting feature for the investor is revealed in this figure: for some quite low volatility portfolios, the third L-moment could be strongly improved with second (and fourth) L-moment(s) only marginally deteriorated. The non-linear non-monotonic discontinuous relations found between L-moment₃ and other L-moments for efficient portfolios (see also Figure 7) are consistent with the relations presented by Athayde and Flôres (2004), Jurczenko, Maillet and Merlin (2006), and Maringer and Parpas (2008), but differ from the relation found in Harvey, Liechty, Liechty and Müller (2002)³².

- Please, Insert Figures 2 to 7 somewhere here -

The investigation of the various two-moment trade-offs between L-moments leads to the following preliminary conclusions. First, the strong link found between the even L-moments seems to be in accordance with the traditional two-moment analysis and can be thought of as rational (even if not automatic - see Brockett and Garven (1998), and Haas (2007)). Since the even L-moments of efficient portfolios are thus found to be in correlation, the temperance of agents (sensitivity to the fourth moment) is closely linked with their risk aversion (sensitivity to the second moment). As seen in Jurczenko and Maillet (2004) in the context of option pricing with higher-moments and as mentioned by Galagedera and Maharaj (2008) examining an extended higher comoments conditional CAPM with realized variables, the (co-)kurtosis seems not to bring a lot of information into the analysis when the (beta) second moment is already in (at least for the type of assets and data sample considered here). Secondly, as already pointed out in the literature, the relation between odd moments is proved unclear. Indeed, the relation between first and

³¹The discontinuities are here reinforced by the imposed no-short sale constraint, and also, to an even greater extent, by the fact that we perform here a large scale optimization (see Athayde and Flôres (2002)).

³²This difference could be explained by a database bias (four equity stocks are used and one of them dominates the others in the first and third moments).

third moments for the efficient portfolios is shown to be non-monotonic (see Post, Vliet and Levy (2008), and Crainich and Eeckhoudt (2008)). Extreme (highest or lowest) greediness of investors (highly sensitive to the first moment) goes with a lower prudence (sensitivity to the third moment), whilst extremely prudent agents will choose portfolios with mild (non extreme) characteristics in terms of first, second and fourth moments. Thirdly, as a consequence, we may say that the prudence characteristic of the investor is crucial when determining her asset allocation, and that taking into account the third-moment might lead to favorable improvements of the utility of the investor, only causing small changes in the three other moment characteristics of optimal portfolios. We further investigate, hereafter, these preliminary conclusions, not only dealing with moments two-by-two as previously, but taking into account all moments at the same time, and focusing more on higher moments than in the above analysis.

We represent in Figure 8 the whole set of the four-moment efficient portfolios in the traditional plane of Markowitz. All portfolios in this Figure are said to be (potentially) Pareto efficient since one may find a rational agent, exhibiting a specific set of marginal rates of substitution between L-moments, that chooses one of these portfolios without any hope of simultaneous improvements in all moments going with another choice. All these portfolios, which do not belong to the Mean-L-moment₂ frontier anymore, show either a high L-moment₃ or a low L-moment₄ amongst all possible portfolios, or a compromise between the two. Color shadings represent either the third or the fourth L-moments respectively on the left and right sub-figures. Three pieces of complementary information to the previous analysis should be mentioned here. First, almost all the interior of the two-moment frontier is full of four-moment efficient portfolios, except the right bottom part of the Figure 8 that does not contain any optimal portfolios: almost any compromise between moments is here reachable, except low mean returns accompanied by high second, third and fourth moments (which are dominated by a better mean portfolio). Secondly, we do not observe a clear link between moments when enlarging the analysis in the four dimensions. For instance, minimization of the second L-moment from high levels (right part of the figure) results in various levels of other moments. Thirdly, and surprisingly, agents who are extremely prudent (paying high attention to the third L-moment) will choose portfolios in the same region (characterized by high-mean, high-volatility, high-skewness and also high-*kurtosis* portfolios) than the agents having the lowest temperance (almost insensitive to the *kurtosis*). But we also remark that choosing amongst these portfolios represents a large (probably unrealistic) price in terms of expected return. In that sense, diverging too far from the two-moment only optimization paradigm (looking

for interesting higher moments) has a cost, which may be seen as almost unreasonable by most investors. However, and once again, we also see portfolios, near the traditional frontier (upper limb), that exhibit better third L-moments. The third and fourth moments of the Mean-L-moment₂ frontier portfolios could thus probably be improved, with only a marginal deterioration of the first two L-moments.

- Please, Insert Figure 8 somewhere here -

We highlight this last fact through Figure 9 where we plot the frontiers of Mean-L-moment₂ efficient portfolios for various given levels of L-moment₃, together with the global mean-L-moment₂ efficient frontier. The sizes of the dots (altogether with their color shadings here) represent the level of fourth L-moment. We thus obtain various peculiar shapes of frontiers for each level of L-moment₃, that are each time very similar to the two-dimension theoretical *parabola*, with different asymptotes and apexes depending upon the third moment. This result is then comparable to the representations of the Mean-Variance-Skewness presented in Jurczenko and Maillet (2001), Buckley, Saunders and Seco (2008), and Mencía and Sentana (2008) in a restricted three-moment world, and by Jurczenko, Maillet and Merlin (2006), Kerstens, Mounir and Woestyne (2007), or Maringer and Parpas (2008) in a four parameter space. The L-moment₂ efficient portfolios with low volatilities indeed appear to be located on the most negative L-moment₃ parts of the shapes. This highlights the fact that for low and high L-moment₂ targeted portfolios, L-moment₃ and L-moment₄ could be significantly improved by reasonable changes in the L-moment₂, as shown in Table I in the case of low volatility portfolios.

- Please, Insert Figure 9 somewhere here -

Now comes the question of utility implications of an allocation also considering higher moments in a dual representation of the investor allocation problem, which is a difficult question mainly because of the complexity of the problem of optimizing utility functions³³.

³³Several issues indeed arise when optimizing utility functions (see Jondeau and Rockinger (2006a) and Garlappi and Skoulakis (2008)). First, excessive sensitivities of utility functions to parameters often plague optimization procedures. Secondly, when performing multi-objective optimization, five theoretical conditions (such as independence of the sub-objectives) have to be respected to ensure the existence of an optimum (see Keeney and Raiffa (1993)). Thus, the multi-objective optimization problem applied to portfolio choice (see Ehrgott, Klamroth and Schwehm (2004)) has to be rigorously defined. Lastly, as shown by Marschinski, Rossi, Tavoni and Cocco (2007), the parameter uncertainty has strong implications in the determination of optimal solutions.

In a very general context, we can start by using the vectors of L-moments of the potential efficient portfolios for defining regions in which the L-moments of efficient portfolios are almost similar (arbitrarily valuing at the same rate the various L-moments since they are of the same order of magnitude). Figure 10 indeed represents the location of portfolios classified into five clusters that have been built using a Self-Organizing Maps algorithm (see Kohonen (2000) and Appendix 3), based on the vectors of L-moments. Each region in Figure 10 (color shadings on the map) represents a set of similar portfolios in terms of their L-moments; we remark that we can distinguish clear regions of portfolios with similar L-moment characteristics. Cluster 4, for instance, groups together most of the traditional two-moment optimal portfolios, whilst cluster 2 is compounded with the highest skewness and *kurtosis* portfolios. In other words, in such a general framework (with no formal specification in link with an explicit utility function), individuals may pick their optimal portfolios in various rather homogeneous regions (corresponding to several preference and risk prototypes), according, first, to the characteristics of the underlying L-moments (Figure 10) and, second, to the way they value them (Figure 12 below).

- Please, Insert Figure 10 somewhere here -

In Figure 12, we propose to illustrate the optimal choice, now valuing the investors' sensitivities and moments according to some flexible usual representations of utility functions (see Appendix 1). We select and represent, from the set of the optimal Four L-moment Efficient Portfolios, the ones that maximize the level of utility for various parameters in relevant ranges, for a Quartic utility function computed with L-moments (imposing here the sensitivities to moments to decrease with the order of the moment) and a Quartic (CRRA) utility function (for plausible reported values of the risk aversion coefficient - see Jondeau and Rockinger (2006a)³⁴). We also generalized our results (bottom figures) using a more flexible Power-exponential utility function. In almost all cases, selected efficient

³⁴As an alternative, we also considered a fourth-order Taylor expansion of the CARA (for Constant Absolute Risk Aversion) exponential utility function, that has also been widely studied in the literature. Yet, it should be noticed, as written in Jondeau and Rockinger (2006a), that we found the same basic results with both types of utility functions.

portfolios are lying on (or not far from) the traditional two-moment frontier.³⁵ We indeed converge to the previous reported results concerning the accuracy of a second-moment approximation, since the four-moment *optima* all lie on the traditional two-moment frontier. As in Simaan (1993) these results suggest, furthermore, that the opportunity cost of the static mean-variance investment strategy is empirically irrelevant in our case. In other words, either the considered class of the utility function is not general enough and/or the departures from Gaussianity and non-linearities present in our database on the period are not large enough to significantly impact the traditional asset allocation.

- Please, Insert Figure 12 somewhere here -

This fact is strengthened by the results presented in Table I and Figure 13. As already mentioned, most of the original stock return series cannot indeed be considered as Normal (see Table I). The same is also observed regarding the optimized portfolio returns. The Jarque-Bera, Anderson-Darling and Lilliefors tests highly reject the Gaussian *hypothesis*, whilst the Kolmogorov-Smirnov test overall result is more contrasted. The two former tests mostly evaluate differences in the tails of the distributions, while the Lilliefors is especially sensitive to distribution gaps located at the mode. For the Kolmogorov-Smirnov test, only the size of the largest difference counts. Differences between empirical probability density functions and the theoretical Gaussian-benchmark ones are then mainly differences in the skewness and the *kurtosis*, small differences in the tails and in the centers of distributions, more likely than large differences somewhere specific. It is fair to mention here that most of the series - original stock and various optimized portfolio returns - clearly differ from the ideal Gaussian *hypothesis* of the Markowitz' model. Our negative result concerning the impact of higher moments on the utility-based optimal choice is not due to a hypothetical (almost-)Gaussianity of the underlying series. What could happen now if we were to intensify the original values of the skewness and the *kurtosis* (keeping the two first moments unchanged)? In Figure 13, we select and represent, from the set of all directionally optimized portfolios, those that are optimal for one of the various Quartic

³⁵Despite the widespread approach in the investor preference literature being to scale the initial agent wealth to one (see Jondeau and Rockinger (2006a)), we also tested *per* curiosity optimal utility-linked choices considering some levels of wealth (for intensifying the role of the higher moments). Our first results were not conclusive in the sense that we found no clear impact on the moment preferences (when sticking to realistic decreasing order sensitivities). Moreover, utility function approximations became in this case far more complicated, since terms appear to concern the dependence upon higher moments (due to the presence of various wealth-moment interactions).

Utility functions - making their higher moments more attractive by a simple linear transformation (multiplying the third L-moment and dividing the fourth L-moment by 2 and 4, with the first and second L-moments unchanged³⁶). In the left (right) figure, the bold grey line represents the Quartic efficient frontier with no change (original data), whilst the thin black line represents the efficient frontier when the third moments of underlying portfolios are multiplied by 2 (respectively by 4), the bold grey dotted line corresponds to the efficient frontier when the fourth moments are divided by 2 (by 4), and the thin black dotted line is related to the efficient frontier when both third and fourth L-moments are (respectively) multiplied and divided by 2 (by 4). We clearly see that, even for unreasonable changes in the higher L-moments (a 100% increase), the optimal portfolios still lie close to the traditional frontier, whilst it is just for a drastic unrealistic intensification that some optimal portfolios start to go away from the usual frontier (but only for some low-medium volatility portfolios). This clearly illustrates that even if we greatly intensify the non-normal features of the data, no clear impact could be highlighted in terms of utility improvements.

- Please, Insert Table I and Figure 13 somewhere here³⁷ -

This last conclusion, considering our pure equity sample in the period 2001-2006 and some classical higher-moment utility functions, so greatly moderates the previous ones when we only considered the higher moment portfolio characteristics without valuing them through a precise utility function. Whatever indeed the truncation order of the utility function chosen, the accuracy of the approximation of the expected utility is still definitely an empirical issue (see Hlawitschka (1994)). Several authors (such as, for instance, amongst others: Markowitz (1991), and Hsieh and Fung (1999) - see Jurczenko and Maillet (2006a), and Jondeau and Rockinger (2006a), for larger lists of references) show using different assets, databases, time periods, constraints, utility functions and parameter sets, that a second-order Taylor expansion already quite accurately approximates the expected utility, leaving us *a priori* with only small potential improvements when dealing with higher

³⁶Since all the two-moment optimal portfolios exhibit neither the highest L-moment₃ nor the smallest L-moment₄ for all levels of L-moment₂, we also tried other simple linear transformations such as multiplying by 2 and 4 (dividing by 2 and 4) both higher L-moments, with no significant change in the conclusion.

³⁷The aim of these representations is to stretch higher moments for making some portfolios even more attractive to investors. Since relations between higher moments are not monotonous, we also tried several intensification functions (see Footnote 36 in the text), with the same general result each time: it is only for large changes that we find a difference between the related efficient frontiers and the traditional ones.

moments (see Jondeau and Rockinger (2003b and 2006a)). These studies confirm the previous research on this subject, since we were not able in a traditional expected utility framework to clearly underline the positive effect of incorporating higher-moments in the asset allocation.

However, a potential application for the investors of the higher L-moments could be in the control of the features of optimal portfolios, since some efficient portfolios, similar in terms of the two first moments, exhibit large differences in higher L-moments as presented in Table II, where we compare some characteristics of second moment alike portfolios. From the set of the Four L-moment Efficient Portfolios with an $L\text{-moment}_2$ that is not significantly different from that of the Minimum $L\text{-moment}_2$ Portfolio (at a 5% threshold using the Log-normal *hypothesis*³⁸ by Alizadeh, Brandt and Diebold (2002) - see Figure 14), we select the Local Maximum Mean, the Local Maximum $L\text{-moment}_3$ and the Local Minimum $L\text{-moment}_4$ Portfolios. The Table II presents the first four L-moments and the Sharpe ratios of the selected local best portfolios and indicates that small (non-significant) changes in the $L\text{-moment}_2$ may lead to some differences in other characteristics, advocating for a potential advantageous control of other features of the traditional optimal portfolios.

- Please, Insert Figure 14 and Table II somewhere here -

IV. Conclusion

We introduce in this article a general method for deriving the set of strong efficient portfolios in the non-convex Four L-moment space, using a shortage optimization function (see Luenberger (1995), Briec, Kerstens and Lesourd (2004), Jurczenko, Maillet and Merlin (2006), and Briec, Kerstens and Jokung (2007)) and a set of robust statistics called L-moments. In this framework, the portfolio efficiency is simply evaluated by looking simultaneously for L-variance and *L-kurtosis* contractions, and mean and (positive) L-skewness expansions. We thus approximate the true but unknown Four L-moment Ef-

³⁸We first tested the hypothesis of Log-normality of the second L-moment using one-year rolling sample volatility of returns on the global Minimum $L\text{-moment}_2$ Portfolio, and the resulting P-statistics of Jarque-Bera, Kolmogorov-Smirnov, Liliefors and Anderson-Darling tests were, respectively, .14, .13, .14, .14, leading to a non-rejection of the null. Using a Fisher test, we then considered the set of portfolios having a return standard deviation not significantly different from the one of the return on the global Minimum $L\text{-moment}_2$ Portfolio. We, secondly, considered portfolios non-significantly different in $L\text{-moment}_2$, using an empirical cumulative density function; overall results were similar.

efficient Frontier by a non-parametric portfolio frontier, using an efficiency measure that guarantees a global optimum in a four-dimensional space. In addition, the shortage function approach can adapt itself to any particular multi-moment asset allocation focusing on return-maximization, (L)-skewness-maximization, (L)-variance-minimization, and (L)-*kurtosis*-minimization. Furthermore, dual interpretations are available without imposing any extra simplifying hypotheses (see Briec, Kerstens and Jokung (2007)).

An empirical application provides a four-dimensional representation of the primal non-convex four L-moments efficient portfolio frontier and illustrate the computational tractability and relative robustness of the approach. The Efficient Frontier estimation recovered from a sample universe of 162 European stocks shows various rational trade-offs between moments that are coherent with previous approaches (see Athayde and Flôres (2004), Jurczenko, Maillet and Merlin (2006), and Maringer and Parpas (2008)). Nevertheless, comparisons between approximations of traditional CRRA, Quartic and Power-expo Utility functions based on a fourth-order Taylor expansion, computed with C- or L-moments (with decreasing absolute values of sensitivities to moments), show that optimal portfolios mainly lie on the conventional mean-variance efficient frontier in a traditional expected utility setting. In other words, we conclude as Jondeau and Rockinger (2003b and 2006a) that higher moments can only matter when investors exhibit major preferences for higher moments and when underlying assets are massively non-Gaussian, that is when we depart a lot from the traditional analysis by Markowitz (1952). However, we also advocate for the use of local optimal higher L-moment efficient portfolios that may have for the investors some advantages over the traditional mean-variance solutions (see also Darolles, Gouriéroux and Jasiak (2008)).

A first natural extension of this study will be to apply the proposed method to non-linear asset payoffs of protected, option-like or alternative investments, where divergences from Normality are truly large and have specific forms (large pointwise discrepancies, and not only, as in our case study, small diluted differences). A second straightforward potential extension is to be found in a conditional version of our four L-moment asset allocation model. A conditional time-varying modelling of the Four L-moments can be achieved either by using a multivariate GARCH-based model specification for the asset return covariances (see Jondeau and Rockinger (2003a and 2006b), and Jondeau, Poon and Rockinger (2007)), or within a robust dynamic autoregressive quantile model (see the CAViaR model by Engle and Manganelli (2004), the QAR by Koenker and Xiao (2006), the CHARN by Martins-Filho and Yao (2006), the DAQ by Gouriéroux and Jasiak (2008), the CARE and EWQR by Taylor (2008a and 2008b)). A third possible extension of our work would be in the

development of an asset pricing relation, exploiting the potential improvement of our new notation of higher-order moments, that generalizes the Gini-CAPM (see Shalit and Yitzhaki (1989), Okunev (1990), and Benson, Faff and Pope (2003)) in the four L-moment direction, which can be done for instance by imposing some restrictions on the joint asset return distribution (see Shalit and Yitzhaki (2005)), and possibly estimated with Realized (L-)comoments (see Galagedera and Maharaj (2008)). Finally, it would be of great interest to investigate the relative performance of our Four L-moment approach with respect to alternative robust non-parametric (see Ledoit and Wolf (2003 and 2004), Kim and White (2004), and Gouriéroux and Liu (2006)) and parametric (Adcock (2008), Buckley, Saunders and Seco (2008), Mencía and Sentana (2008), Cvitanić, Polimenis and Zapatero (2008)) extended multi-moment asset allocation and pricing models in an international setting (see Guidolin and Timmerman (2008)).

A Appendix 1

We briefly present below the main utility functions and their approximations based on the first four moments used in Figure 12.

Let $U(.)$ be a general utility function of the end-of-period wealth W . A Taylor series expansion allows us to obtain the expected utility (see Jondeau and Rockinger (2006a), Jurczenko and Maillet (2006a), and Garlappi and Skoulakis (2008) for precise conditions³⁹ for the development to be exact or approximative), written as such:

$$E[U(W)] = U(\bar{W}) + U^{(1)}(\bar{W}) E[W - \bar{W}] + 2^{-1} U^{(2)}(\bar{W}) E[(W - \bar{W})^2] + (3!)^{-1} U^{(3)}(\bar{W}) E[(W - \bar{W})^3] + (4!)^{-1} U^{(4)}(\bar{W}) E[(W - \bar{W})^4], \quad (42)$$

where $E(.)$ is the expectation operator, $U^{(n)}$, for $n = [1, \dots, 4]$, corresponds to the n -th derivative of the utility function with respect to her final wealth denoted W , with W_0 the

³⁹Even if the Taylor series expansion is a standard and a well established way to express utility functions in terms of moments (see Samuelson (1970)), it is to be noticeable that all moment preferences (first derivatives) only depend upon the first moment of return (alone). This feature is not merely intuitive since an investor targeting a high return portfolio would be more sensitive to extreme events compared to an investor choosing a less profitable portfolio. This claims for a more general utility representation than the usual ones we tested here. In addition, Loistl (1976) argues that such approximations could also be largely biased if some conditions are not respected (see Garlappi and Skoulakis (2008) for details on these conditions).

initial wealth (equal to the unity by simplification⁴⁰), $\overline{W} = E(W) = [1 + E(R_p)]$ the expected final wealth, $R_p = W/W_0 - 1$ the random return on the portfolio held by the investor.

Thus, if we consider a Constant Relative Risk Aversion Utility function (see Mehra and Prescott (1985)) such as (with previous notations):

$$U(W) = (1 - \gamma)^{-1} W^{1-\gamma}, \quad (43)$$

we obtain using C-moments:

$$E[U(W)] = (1 - \gamma)^{-1} (m^1)^{1-\gamma} - 2^{-1} \gamma m^2 (m^1)^{-\gamma-1} + (3!)^{-1} \gamma (\gamma + 1) m^3 (m^1)^{-\gamma-2} - (4!)^{-1} \gamma (\gamma + 1) (\gamma + 2) m^4 (m^1)^{-\gamma-2}, \quad (44)$$

with $\gamma =]0, 20] \setminus \{1\}$ and where the m^n , for $n = [1, \dots, 4]$, are the first four centered moments of the returns.

If we now consider a Quartic Utility function (see Jondeau and Rockinger (2006a) and Jurczenko and Maillet (2006a)) as such (with previous notations):

$$U(W) = \beta_0 + \beta_1 W - \beta_2 W^2 + \beta_3 W^3 - \beta_4 W^4, \quad (45)$$

we have under the same restrictions:

$$E[U(W)] = a_0 + a_1 m^1 - a_2 (m^1)^2 + a_3 (m^1)^3 + a_4 (m^1)^4 + [a_2 + 3a_3 m^1 + 6a_4 (m^1)^2] m^1 + [a_3 + 4a_4 m^1] m^3 + a_4 m^4. \quad (46)$$

⁴⁰ A well-known criticism of portfolio choice models is the absence of any wealth effect in the analysis (see Quizon, Binswanger and Machina, (1984)); some authors suggest that the relative risk aversion increases with wealth, whilst others conclude with the opposite effect (see Peress, (2004)). Nevertheless, the main approach in the investor preference literature is to scale the initial agent wealth to one. We first thought that such an approach could be very restrictive and, accordingly, we also tested optimal utility-linked choices considering various levels of wealth (as an attempt to intensify the role of the higher moments). Our first results were not, however, conclusive. More precisely, we did not find any clear impact on the moment preferences. Added to the fact that the utility function approximations become more complicated in terms of dependence to the higher moments (due to the presence of various wealth-moment interactions), we, secondly, chose to rescale to one the initial wealth for the sake of simplicity. This renormalization allows us to write simply the expansion and to easily make the link between moments and preferences in the context of series expansion. It ensures also the complete equivalence between the expected utility expressions both in terms of return and terminal wealth - with a simple substitution of W (in this Appendix and in Figure 12) by R_p (in the *corpus* of the text). See Brckett and Golden (1987) and Jondeau and Rockinger (2006a) for discussions regarding completely monotone utility functions and independence to wealth levels (see also Footnote 35 in the text).

where, for $n = [1, \dots, 4]$, the m^n being the first four centered moments of the returns and $a_i \in IR$.

If we finally consider a Power-exponential Utility function (see Saha (1993), and Holt and Laury (2002)) as such (with previous notations):

$$U(W) = \alpha^{-1} [1 - \exp(-\alpha W^{1-\gamma})], \quad (47)$$

we now have using C-moments:

$$E[U(W)] = \beta_1 + 2^{-1}\beta_2 m^2 + (3!)^{-1}\beta_3 m^3 + (4!)^{-1}\beta_4 m^4, \quad (48)$$

where, for $n = [1, \dots, 4]$, the m^n being the first four centered moments of the returns and the $\beta_n \in IR$ are such as:

$$\begin{cases} \beta_1 = \alpha^{-1} \{1 - \exp[-\alpha(m^1)^{1-\gamma}]\} \\ \beta_2 = (1 - \gamma) \exp[-\alpha(m^1)^{1-\gamma}] \times [-\gamma(m^1)^{1-\gamma} - \alpha(1 - \gamma)(m^1)^{-2\gamma}] \\ \beta_3 = (1 - \gamma)^2 \exp[-\alpha(m^1)^{1-\gamma}] \times [2\alpha\gamma(m^1)^{-1-2\gamma} + \alpha^2(1 - \gamma)(m^1)^{-3\gamma} \\ \quad - \gamma(m^1)^{-\gamma} + \alpha(m^1)^{-1-2\gamma}] \\ \beta_4 = (1 - \gamma)^2 \exp[-\alpha(m^1)^{1-\gamma}] \times \\ \quad [2\alpha\gamma(2\gamma - 1)(m^1)^{-2-2\gamma} - 5\alpha^2\gamma(1 - \gamma)(m^1)^{-1-3\gamma} + \gamma^2(m^1)^{-\gamma-1} \\ \quad \alpha(-\gamma^2 - \gamma + 1)(m^1)^{-2\gamma} - \alpha^3(1 - \gamma)^2(m^1)^{-4\gamma} - \alpha^2(1 - \gamma)(m^1)^{1-3\gamma}], \end{cases}$$

with $\alpha \in IR \setminus \{0\}$, $\gamma \in IR \setminus \{1\}$, $\alpha(1 - \gamma) > 0$, where α and $(1 - \gamma)$ respectively govern the Relative and Absolute Risk Aversions. ■

B Appendix 2

We hereafter recall the propositions given in the *corpus* of the article regarding the optimization programs and present their proofs.

Proposition 1. *For every $\mathbf{g} \in (IR_+ \times IR_- \times IR_+ \times IR_-) \setminus \{0\}$ and every $\mathbf{w} \in \mathfrak{F}$, there exists a unique element δ^* , $\delta^* \in IR_+$, such that:*

$$S_{\mathbf{g}}(\mathbf{w}) = \lambda_{\mathbf{w}} + \delta^* \mathbf{g}. \quad (34)$$

Proof of Proposition 1.

First, we recall that the set of the feasible portfolios \mathfrak{A} can be expressed as follows:

$$\mathfrak{A} = \left\{ \mathbf{w} \in IR^N : \mathbf{w}' \mathbf{1} = 1 \text{ and } \mathbf{w} \geq \mathbf{0} \right\}, \quad (26)$$

where \mathbf{w}' is the $(1 \times N)$ transposed vector of the investor's holdings in the various risky assets, $\mathbf{1}$ is the $(N \times 1)$ unitary vector and $\mathbf{0}$ is the $(N \times 1)$ null vector. Also recall that the set of the feasible portfolios in the four L-moment space in a free disposal representation, denoted \mathfrak{F} (see Briec, Kerstens and Lesourd (2004), and Briec, Kerstens and Jokung (2007)) writes:

$$\mathfrak{F} = \{\boldsymbol{\lambda}_{\mathbf{w}} : \mathbf{w} \in \mathfrak{A}\} + [(-IR_+) \times IR_+ \times (-IR_+) \times IR_+], \quad (27)$$

where $\boldsymbol{\lambda}_{\mathbf{w}}$ is the (4×1) vector of the first four L-moments of the portfolio return $R_{\mathbf{w}}$, *i.e.*:

$$\boldsymbol{\lambda}_{\mathbf{w}} = [\lambda_1(R_{\mathbf{w}}); \lambda_2(R_{\mathbf{w}}); \lambda_3(R_{\mathbf{w}}); \lambda_4(R_{\mathbf{w}})]'.$$

Secondly, we recall the definition of the (generic) partial order relation, denoted by \succeq , on IR^4 , that is for any $(\boldsymbol{\lambda}, \tilde{\boldsymbol{\lambda}}) \in (IR^4)^2$:

$$\boldsymbol{\lambda} \succeq \tilde{\boldsymbol{\lambda}} \iff [\lambda_1 \geq \tilde{\lambda}_1, \lambda_2 \leq \tilde{\lambda}_2, \lambda_3 \geq \tilde{\lambda}_3, \lambda_4 \leq \tilde{\lambda}_4], \quad (28)$$

altogether with a strict relation, denoted by \succ , as such:

$$\boldsymbol{\lambda} \succ \tilde{\boldsymbol{\lambda}} \iff [\lambda_1 > \tilde{\lambda}_1, \lambda_2 < \tilde{\lambda}_2, \lambda_3 > \tilde{\lambda}_3, \lambda_4 < \tilde{\lambda}_4]. \quad (29)$$

Thirdly, for every $\mathbf{w} \in \mathfrak{A}$ and for every $\mathbf{g} \in (IR_+ \times IR_- \times IR_+ \times IR_-) \setminus \{0\}$, let us define the set $\mathfrak{D}_{\mathbf{w}, \mathbf{g}}$ of admissible distances from the portfolio \mathbf{w} to the weakly efficient frontier in the direction \mathbf{g} , such as:

$$\mathfrak{D}_{\mathbf{w}, \mathbf{g}} = \{\delta \in IR_+ : (\boldsymbol{\lambda}_{\mathbf{w}} + \delta \mathbf{g}) \in \mathfrak{F}\}. \quad (49)$$

Let now the mapping $\Lambda(\cdot)$ from \mathfrak{A} to IR^4 defined by, for every $\mathbf{w} \in \mathfrak{A}$:

$$\Lambda(\mathbf{w}) = \boldsymbol{\lambda}_{\mathbf{w}}. \quad (50)$$

The mapping $\Lambda(\cdot)$ is continuous since all coordinate functions are polynomial functions. Moreover, since \mathfrak{A} is a compact set, then $\Lambda(\mathfrak{A})$ is also a compact set (see Theorem 4.14, p.89, in Rudin (1976)). We thus have:

$$\mathfrak{F} = \Lambda(\mathfrak{A}) + [(-IR_+) \times IR_+ \times (-IR_+ \times IR_+)]. \quad (51)$$

So there exists $\bar{\boldsymbol{\lambda}}_{\mathbf{w}} \in IR^4$ such that for every $\boldsymbol{\lambda}_{\mathbf{w}} \in \mathfrak{F}$, $\bar{\boldsymbol{\lambda}}_{\mathbf{w}} \succeq \boldsymbol{\lambda}_{\mathbf{w}}$. This implies that $\mathfrak{D}_{\mathbf{w}, \mathbf{g}}$ is bounded.

The set \mathfrak{F} is the sum of a compact set and a closed set, so it is a closed set. Moreover, the mapping $\delta \mapsto (\lambda_{\mathbf{w}} + \delta \mathbf{g})$ is continuous on IR_+ , and then $\mathfrak{D}_{\mathbf{w}, \mathbf{g}}$ is closed. We deduce that $\mathfrak{D}_{\mathbf{w}, \mathbf{g}}$ is also a compact. Then, there exists $\delta^* \geq 0$ such that $S_{\mathbf{g}}(\mathbf{w}) = (\lambda_{\mathbf{w}} + \delta^* \mathbf{g})$.

At this point, we note, however, that the mapping $\Lambda(\cdot)$ is not necessarily bijective (depending on the considered set of portfolio characteristics), meaning that several different portfolios may potentially be located on the same point of the efficient frontier. In order to avoid this uncertainty, we can also think in terms of equivalence classes of portfolios, formally defining the equivalence relation on the set of portfolios by:

$$\mathbf{w} \sim \check{\mathbf{w}} \iff \Lambda(\mathbf{w}) = \Lambda(\check{\mathbf{w}}), \quad (52)$$

for every \mathbf{w} and $\check{\mathbf{w}}$ in \mathfrak{A} .

Then, the equivalence class associated to a portfolio \mathbf{w} , and denoted $\dot{\mathbf{w}}$, is:

$$\dot{\mathbf{w}} = \{\check{\mathbf{w}} \in \mathfrak{A} \mid \mathbf{w} \sim \check{\mathbf{w}}\}, \quad (53)$$

where the sign \sim stands for the equivalence relation expressed in terms of utility for the investor.

We also remark that the fact that a set of L-moments determine a unique density function (on the contrary of C-moments, see the so-called Hamburger problem - Cf. Hamburger (1920), Jondeau and Rockinger (2003a), and Jurczenko and Maillet (2006a)) allows us to make the link between the weights of a portfolio, the density of returns on it, its L-moments and the utility function of the investor (provided that the density function can be perfectly described by the first four L-moments, which is the case for a large family of four parameter densities). For the sake of simplicity, however, we considered in the *corpus* of the text the term portfolio weights, even if, strictly speaking, we should have referred to equivalence classes of these portfolio weights. ■

Proposition 2. *If $(\mathbf{w}^*, \delta^*) \in (\mathfrak{A} \times IR_+)$ is a local solution of the following non-linear optimization program $\mathfrak{P}_{\mathbf{w}_p, \mathbf{g}}$.*

$$\text{Max}_{(\mathbf{w}, \delta) \in (\mathfrak{A} \times IR_+)} \Phi_{\mathbf{w}_p, \mathbf{g}}(\mathbf{w}), \quad (40)$$

it is then a global solution.

Proof of Proposition 2.

Let us denote:

$$\mathfrak{J} = \{(\mathbf{w}, \delta) \in (\mathfrak{A} \times IR_+) : \lambda_{\mathbf{w}} \succeq (\lambda_{\mathbf{w}_p} + \delta \mathbf{g})\}, \quad (54)$$

which is the hypograph of the mapping $\Phi_{\mathbf{w}_p, \mathbf{g}}(\cdot)$ with the order relation, denoted \succeq and defined above.

We then have:

$$\Phi_{\mathbf{w}_p, \mathbf{g}}(\mathbf{w}) = \sup_{\delta \in IR_+} \{\delta : (\mathbf{w}, \delta) \in \mathfrak{J}\}. \quad (55)$$

Assume that the couple (\mathbf{w}_1, δ_1) constitutes a local maximum, but is not a global one. In that case, there exists a couple $(\mathbf{w}_2, \delta_2) \in \mathfrak{J}$ such that:

$$\delta_2 > \delta_1. \quad (56)$$

This implies that for all $\delta \in [\delta_1, \delta_2]$, $(\mathbf{w}_2, \delta) \in \mathfrak{J}$. Therefore, a neighborhood $\mathcal{N}[(\mathbf{w}_1, \delta_1), \varepsilon]$ where $\varepsilon > 0$, such that $\delta_1 \geq \delta$ for all $(\mathbf{w}, \delta) \in \mathcal{N}[(\mathbf{w}_1, \delta_1), \varepsilon]$ does not exist. Consequently, if (\mathbf{w}^*, δ^*) is a local maximum, then it is also a global maximum. ■

C Appendix 3

We briefly present herein the data mining technique we applied for grouping together optimal portfolios as illustrated in Figure 10 (and in Figure 11 below). Self-Organizing Maps (SOM or Kohonen Maps) are a clustering method with their roots in Artificial Neural Networks. SOM can be used at the same time both to reduce the amount of relevant data by clustering, and for projecting the data nonlinearly onto a lower dimensional display. Due to its unsupervised learning and topology preserving proprieties, the SOM algorithm has proven to be especially suitable in visual analysis of high dimensional sets. They have already been applied in various fields in general, and in finance in particular, for clustering elements sharing some similarities. With no pretension of exhaustivity, some examples of SOM' financial applications are to be found in Deboeck and Kohonen (1998), Resta (2001), Maillet and Rousset (2003), Das and Das (2004), Moreno, Marco and Olmeda (2006), and Ben Omrane and de Bodt (2007). For further details on this data-mining technique, see Kohonen (2000) and Guinot, Maillet and Rousset (2006).

A direct characterization of (65,253) optimal portfolios based on the four L-moment computations is not an easy task. It is very natural to try to reduce the dimension of the problem by considering an exploratory algorithm such as SOM. Our goal is here to group optimal portfolios according to their first four L-moments and to visualize their various locations relative to the efficient frontier. SOM thus enable us to create homogeneous clusters as well as virtual portfolios representative of each cluster. In the following, we

will consider the $(65, 253 \times 4)$ matrix \mathbf{X} , where each line \mathbf{x} corresponds to the estimated first four L-moments of any optimal portfolio.

In order to define the network, the number of neurons (*i.e.* the number of clusters) as well as the shape of the grid (a string or a lattice generally) have to be specified. Let us consider K the number of neurons (units or “code vectors”) and $I = [1, 2, \dots, K]$ the total set of neurons.

The network state at the iteration j is given by:

$$\mathbf{V}(j) = [\mathbf{V}_1(j), \mathbf{V}_2(j), \dots, \mathbf{V}_K(j)], \quad (57)$$

where $\mathbf{V}_i(j)$ is the weight vector of unit i , its size being equal to the dimension of the data to be classified. In our case, we classify portfolios according to their first four L-moments, each weight vector is thus a 4-dimensional vector.

The network is first randomly initialized (iteration 1) from the input set (with previous notations):

$$\begin{aligned} \mathbf{V}(1) &= [\mathbf{V}_1(1), \mathbf{V}_2(1), \dots, \mathbf{V}_K(1)] \\ &= \left\{ \begin{bmatrix} \hat{\lambda}_1^1(1) \\ \hat{\lambda}_2^1(1) \\ \hat{\lambda}_3^1(1) \\ \hat{\lambda}_4^1(1) \end{bmatrix}, \begin{bmatrix} \hat{\lambda}_1^2(1) \\ \hat{\lambda}_2^2(1) \\ \hat{\lambda}_3^2(1) \\ \hat{\lambda}_4^2(1) \end{bmatrix}, \dots, \begin{bmatrix} \hat{\lambda}_1^K(1) \\ \hat{\lambda}_2^K(1) \\ \hat{\lambda}_3^K(1) \\ \hat{\lambda}_4^K(1) \end{bmatrix} \right\} \end{aligned} \quad (58)$$

where λ_k^i , with $i \in I$ and $k = [1, \dots, 4]$, corresponds to the empirical k -th L-moment of a portfolio randomly drawn from \mathbf{X} .

The SOM algorithm is then recursively defined by the following iterations.

1. Draw randomly another observation \mathbf{x} for the set of optimal portfolio L-moments.
2. Find the associated winning unit $i_w(\mathbf{x}, \mathbf{V})$ also called the Best Matching Unit (noted *BMU*), that is the unit whose weight $\mathbf{V}_{i_w(\mathbf{x}, \mathbf{V})}$ is the closest to input \mathbf{x} (with previous notations):

$$BMU = i_w[\mathbf{x}(j+1), \mathbf{V}(j)] = \underset{(\mathbf{V}_i, i) \in (IR^4 \times I)}{\text{ArgMin}} \{ \|\mathbf{x}(j+1) - \mathbf{V}_i(j)\| \}, \quad (59)$$

where $\|\cdot\|$ is the Euclidian norm.

3. Once the *BMU* is found, the weight vectors of the SOM are updated so that the *BMU* and the activated neighbors are moved closer to the input vector. The SOM update rule is, $\forall i \in I$ (with previous notations):

$$\mathbf{V}_i(j+1) = \mathbf{V}_i(j) - \tau(j) \mathbb{k}_{BMU, i}(j) [\mathbf{V}_i(j) - \mathbf{x}(j+1)], \quad (60)$$

where $\tau(\cdot)$ is the learning rate function, which is a $]0, 1[$ -valued decreasing function. The function $\mathbb{k}_{BMU,i}(\cdot)$ is the neighborhood (Gaussian) kernel around the winner unit BMU , that also decreases with iterations.

Following Guinot, Maillet and Rousset (2006), we choose to apply a robust version of the Kohonen algorithm named “Robust Maps”. Through an intensive use of bootstrap, this enables us to overcome the potential convergence problem of the general algorithm. Also for practical considerations, we choose to classify all optimal portfolios using their empirical first four L-moments, and to group and project them onto an arbitrary string compound of five neurons. Thus, the overall algorithm leads to the creation of five homogeneous clusters of four L-moment efficient portfolios.

Figure 10 in the text is based on the classification of raw estimated L-moments of optimal portfolios. In order to make the picture more clear, and also to mitigate the problem of L-moment differences in scale (which is, however, less critical than for C-moments, since L-moments are all rescaled differences of returns and not of the magnitude of returns at various powers), we also run the algorithm on quantiles (ranking) of the L-moments of optimal portfolios (based on their empirical cumulative distributions). The aim is here to cluster the highest values of the population L-moment of efficient portfolios together, taking into account all first L-moments at the same time. The resulting classification (see Figure 11 below) based on estimated quantiles of L-moments slightly differs from the first classification based on raw L-moments (see Figure 10 in the text). The clusters seem to be more homogeneous (more concentrated), with a clear breakdown operated by the two first L-moments (along the traditional efficient frontier). Thus, the clusters are sequenced by descending order in terms of the first two L-moments: investors willing large mean returns (and accepting large related second L-moments) should choose a portfolio in cluster 1, whilst a more risk averse agent should pick up a portfolio within cluster 5. Focusing now on higher L-moments, it appears that the first cluster groups together portfolios with a high $L\text{-moment}_4$ (and $L\text{-moment}_3$ above zero), whilst the second cluster (in the center of the *parabola*) exhibits a high average $L\text{-moment}_3$. Lowest $L\text{-moment}_4$ portfolios are grouped in the fourth cluster, whereas those with the lowest (negative) third L-moments mostly belong to the fifth cluster. Finally, the third cluster corresponds to portfolios with medium $L\text{-moment}_4$ values. Overall, we can see from this clusterization, based on quantiles of L-moments (rather than on L-moments themselves), that regions in the projected efficient plane may be easily explained by the various L-moment characteristics of the optimal primal efficient portfolios.

- Please, Insert Figure 11 somewhere here -

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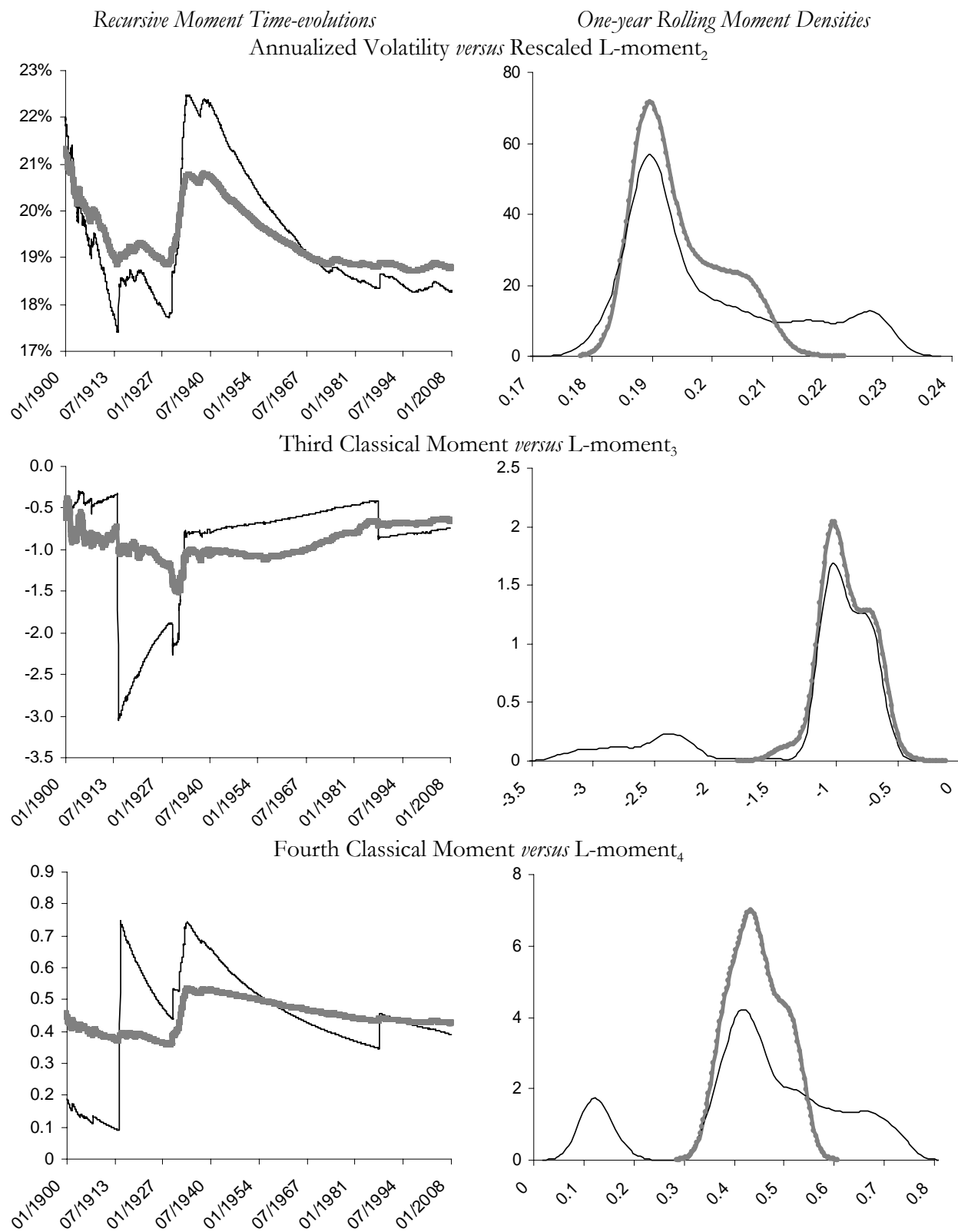


Figure 1. Comparison of Classical Moments *versus* L-moments. Source: *Bloomberg*, Daily Net Asset Values (01/01/1896-01/18/2008) in USD; computations by the authors. On the left side, the y-axis corresponds to the level (times 1.10^6 for the third and fourth moments) of the recursive classical moments (in thin black line); the corresponding recursive L-moments (in grey bold line) are rescaled in order to get the same means as their related Conventional moments. On the right side, all various moments are estimated using a one-year rolling window. The annual daily L-moments are rescaled in order to have the same means as their related C-moments. The nonparametric densities are estimated with a Gaussian kernel using the cross-validation *criterion* (see Silverman (1986)). The probability density function of the classical moments corresponds to thin black lines whilst the one of the L-moments is in bold grey.

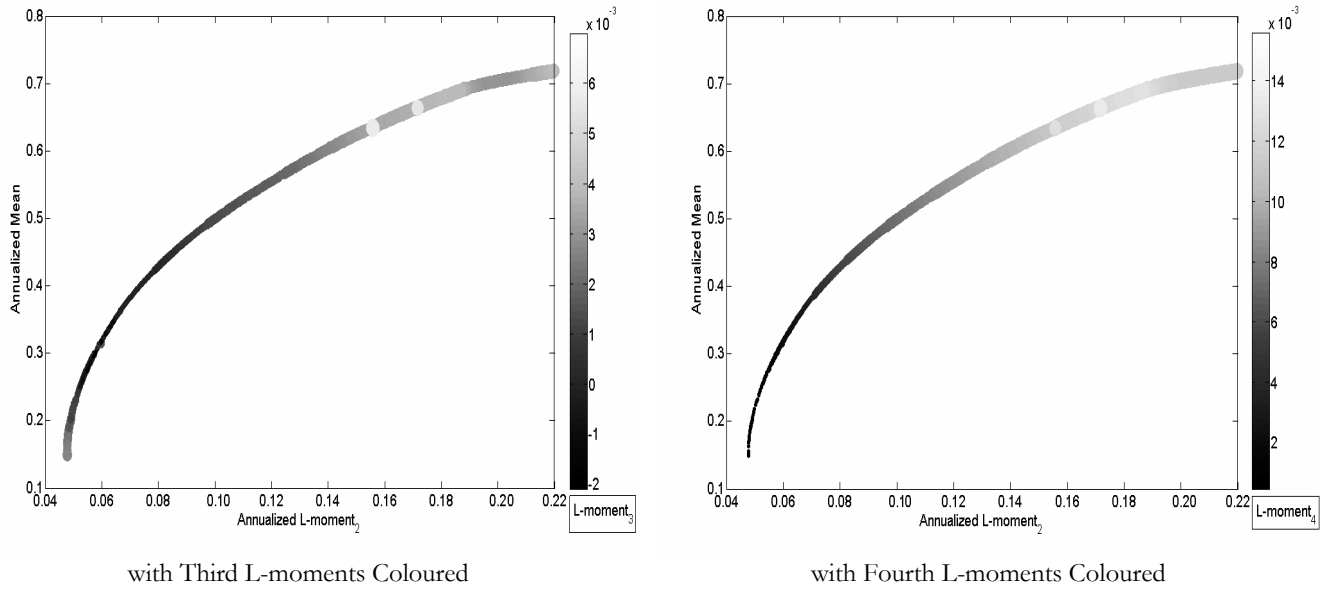


Figure 2. First Four L-moment Constrained Efficient Frontier in the L_1 - L_2 Plane. Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. The constrained efficient frontier is compounded with 2,610 simulated portfolios, obtained after optimization in 9,637 directions of random portfolios built with 162 European equities. Sizes of dots and their colour shadings (from black small low levels to white large high levels) represent (respectively) the third (fourth) L-moments on the left figure (the right figure).

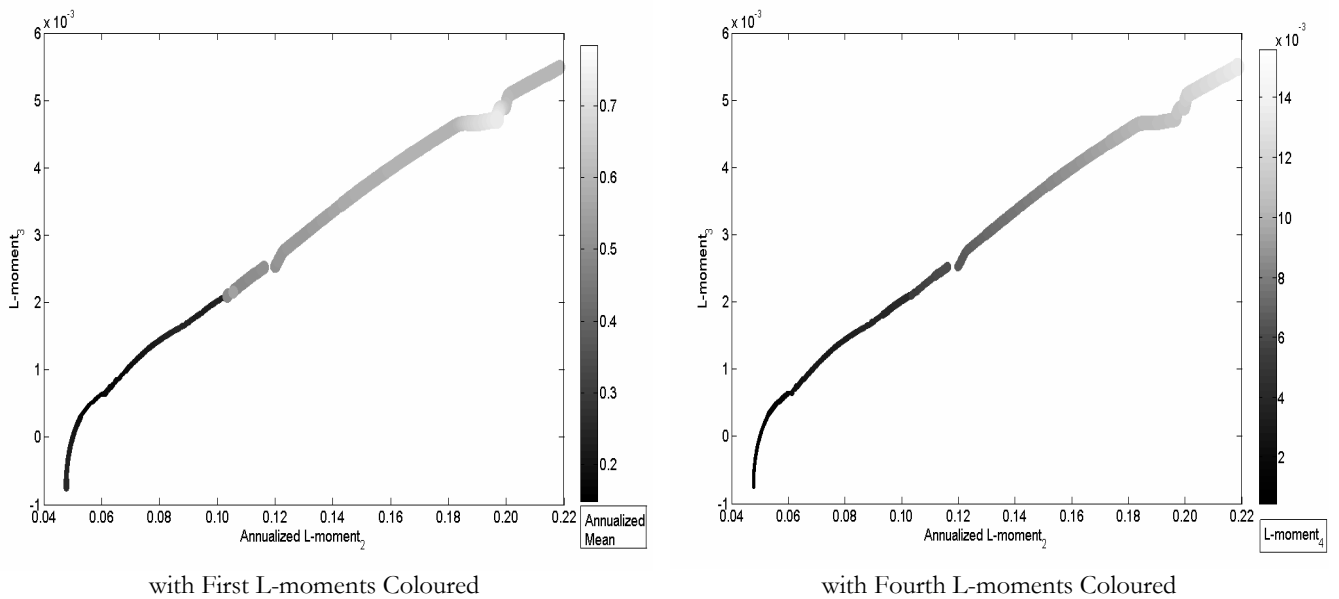


Figure 3. First Four L-moment Constrained Efficient Frontier in the L_2 - L_3 Plane. Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. The constrained efficient frontier is compounded with 1,932 simulated portfolios, obtained after optimization in 9,637 directions of random portfolios built with 162 European equities. Sizes of dots and their colour shadings (from black small low levels to white large high levels) represent (respectively) the first (fourth) L-moments on the left figure (the right figure).

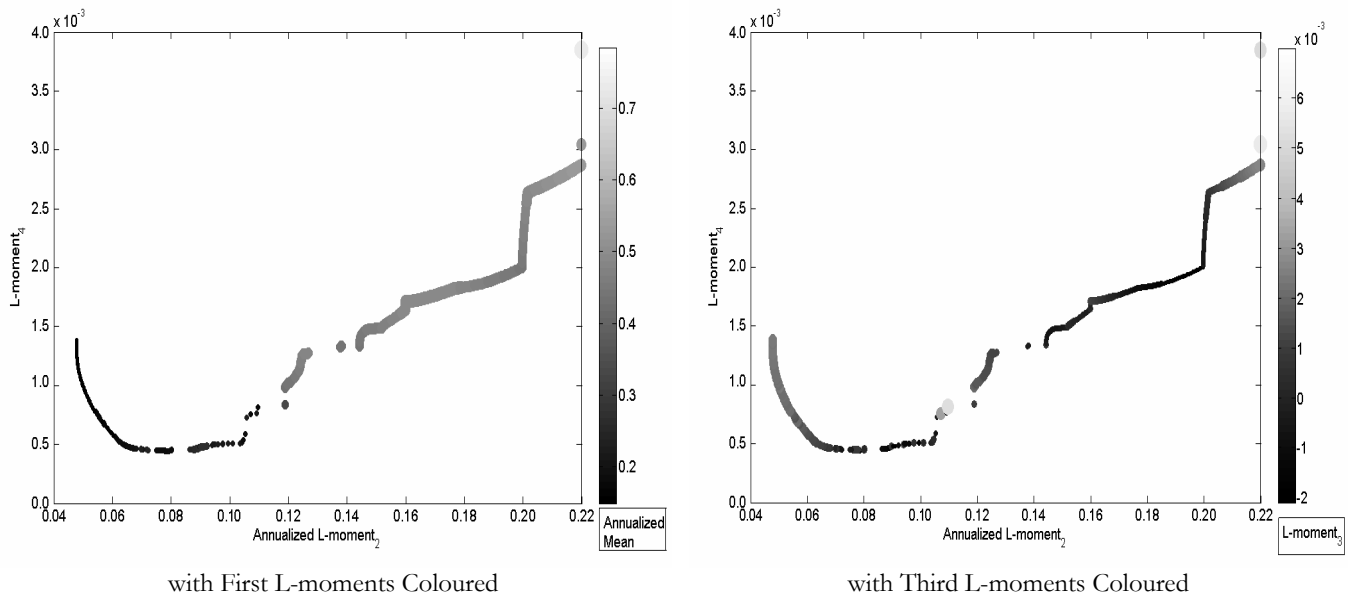


Figure 4. First Four L-moment Constrained Efficient Frontier in the L_2 - L_4 Plane. Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. The constrained efficient frontier is compounded with 731 simulated portfolios, obtained after optimization in 9,637 directions of random portfolios built with 162 European equities. Sizes of dots and their colour shadings (from black small low levels to white large high levels) represent (respectively) the first (third) L-moments on the left figure (the right figure).

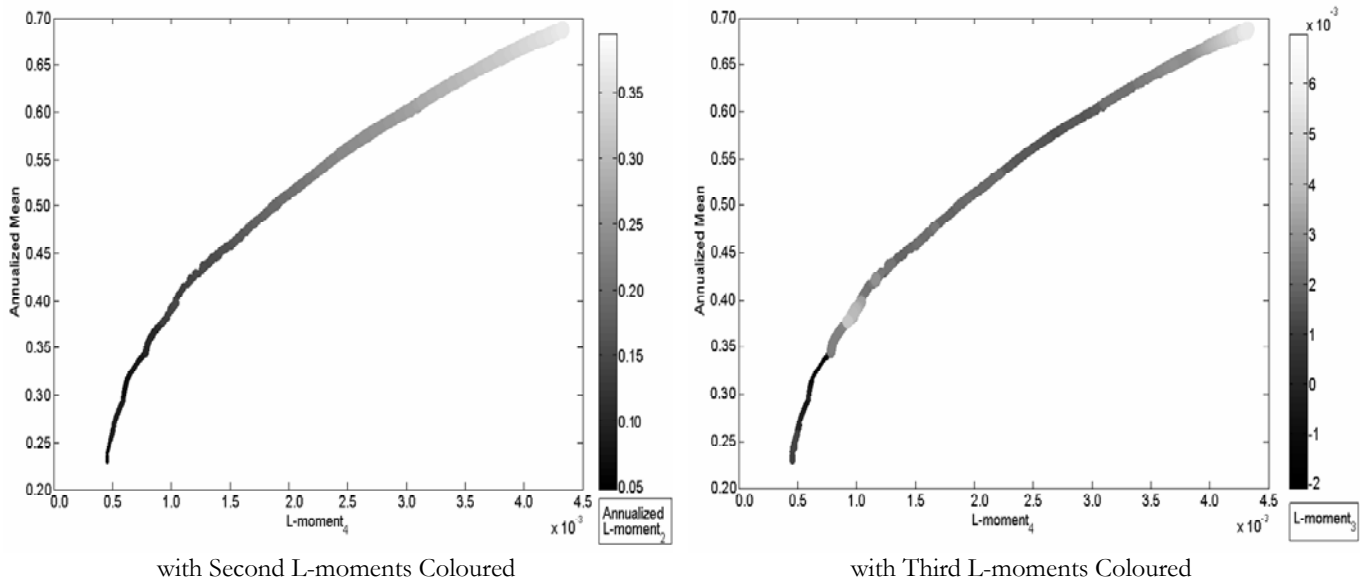


Figure 5. First Four L-moment Constrained Efficient Frontier, in the L_1 - L_4 Plane. Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. The constrained efficient frontier is compounded with 1,161 simulated portfolios, obtained after optimization in 9,637 directions of random portfolios built with 162 European equities. Sizes of dots and their colour shadings (from black small low levels to white large high levels) represent (respectively) the second (third) L-moments on the left figure (the right figure).

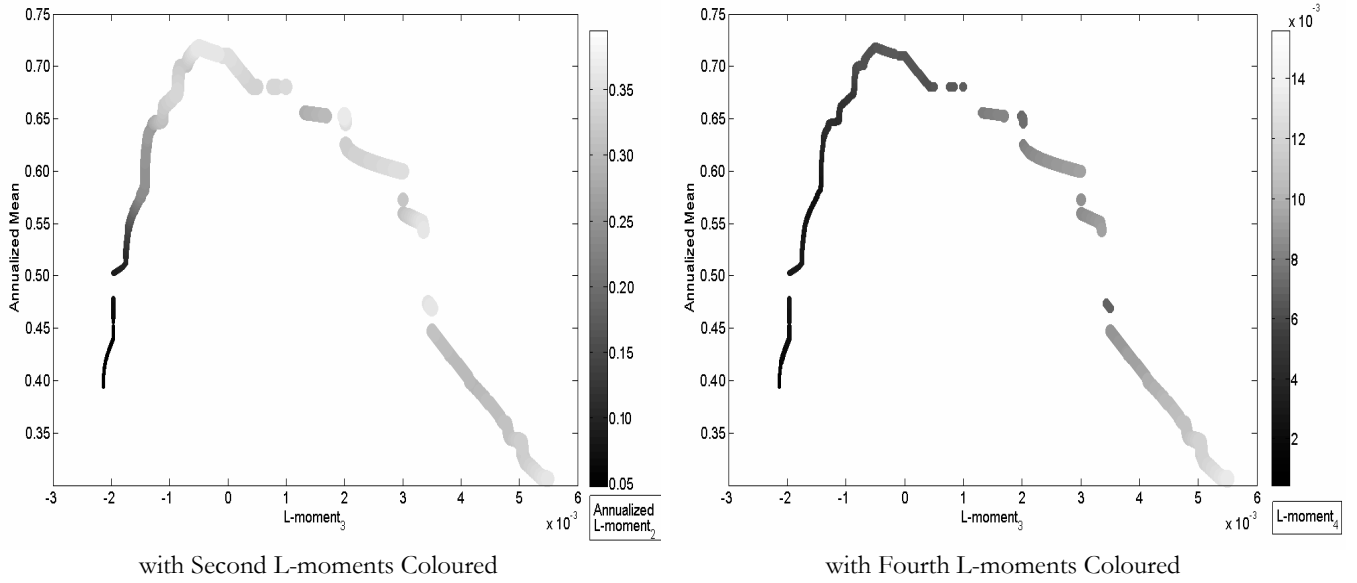


Figure 6. First Four L-moment Constrained Efficient Frontier in the L_1 - L_3 Plane. Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. The constrained efficient frontier is compounded with 1,925 simulated portfolios, obtained after optimization in 9,637 directions of random portfolios built with 162 European equities. Sizes of dots and their colour shadings (from black small low levels to white large high levels) represent (respectively) the second (fourth) L-moments on the left figure (the right figure).

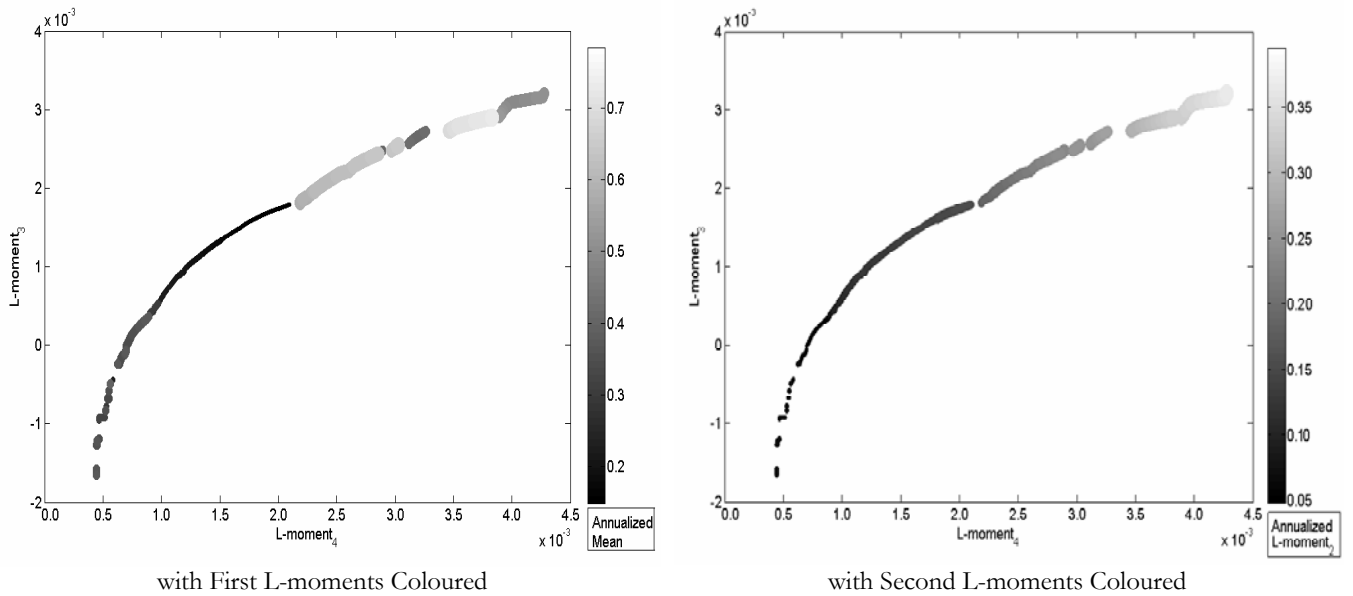


Figure 7. First Four L-moment Constrained Efficient Frontier in the L_3 - L_4 Plane. Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. The constrained efficient frontier is compounded with 1,244 simulated portfolios, obtained after optimization in 9,637 directions of random portfolios built with 162 European equities. Sizes of dots and their colour shadings (from black small low levels to white large high levels) represent (respectively) the first (second) L-moments on the left figure (the right figure).

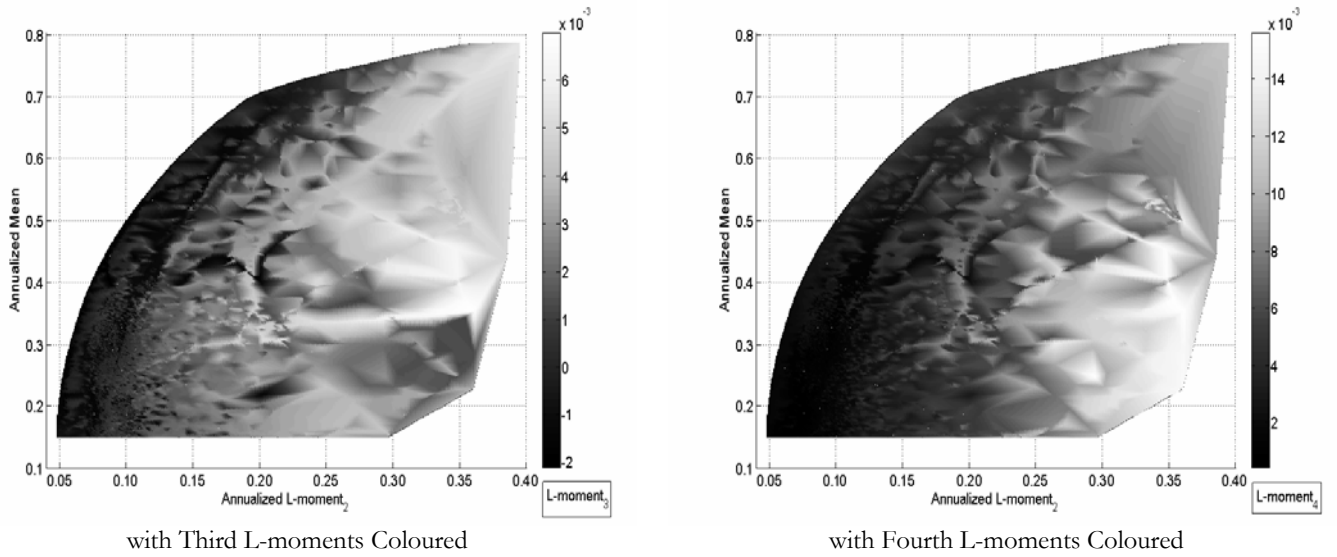


Figure 8. First Four L-moment Constrained Efficient Frontier in the L_1 - L_2 - L_3 Space. Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. The constrained efficient frontier is compounded with 65,253 simulated portfolios, obtained after optimization in 9,637 directions of random portfolios built with 162 European equities. Sizes of dots and their colour shadings (from black small low levels to white large high levels) represent (respectively) the third (fourth) L-moments on the left figure (the right figure).

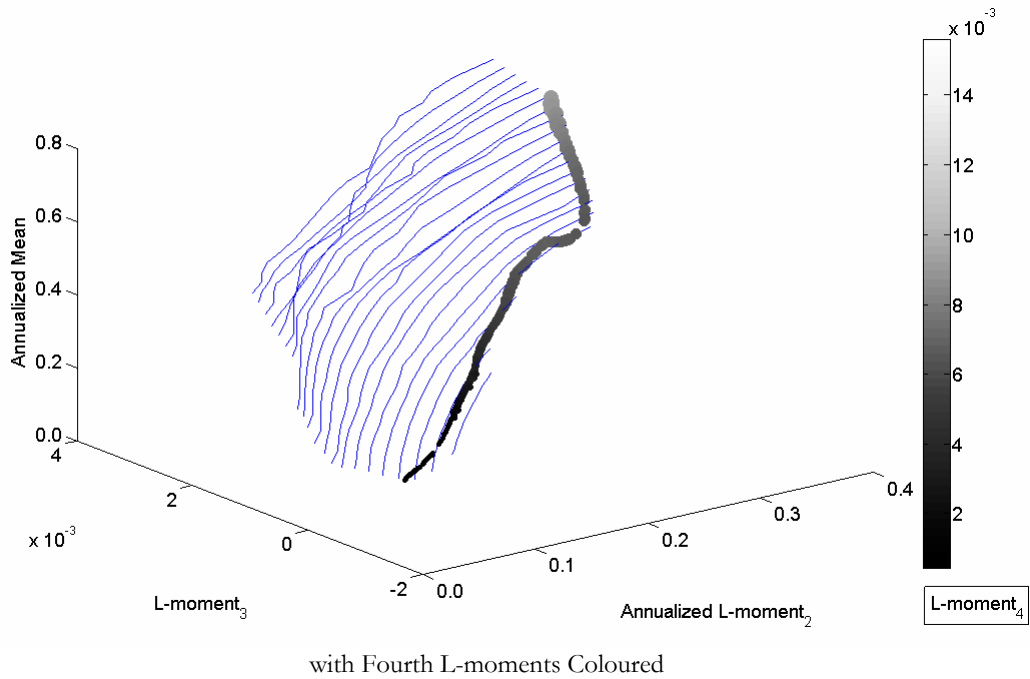


Figure 9. First Four L-moment Constrained Efficient Frontier in the L_1 - L_2 - L_3 Space. Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. The constrained efficient frontier is compounded with 65,253 simulated portfolios, obtained after optimization in 9,637 directions of random portfolios built with 162 European equities. Lines correspond to Mean L-moment2 Efficient Portfolios when the level of L-moment3 is constrained. Sizes of dots and their colour shadings (from black small low levels to white large high levels) represent the Four L-moments of Efficient Portfolios in the Mean-L-moment2 plane (*i.e.* the traditional Efficient Portfolios).

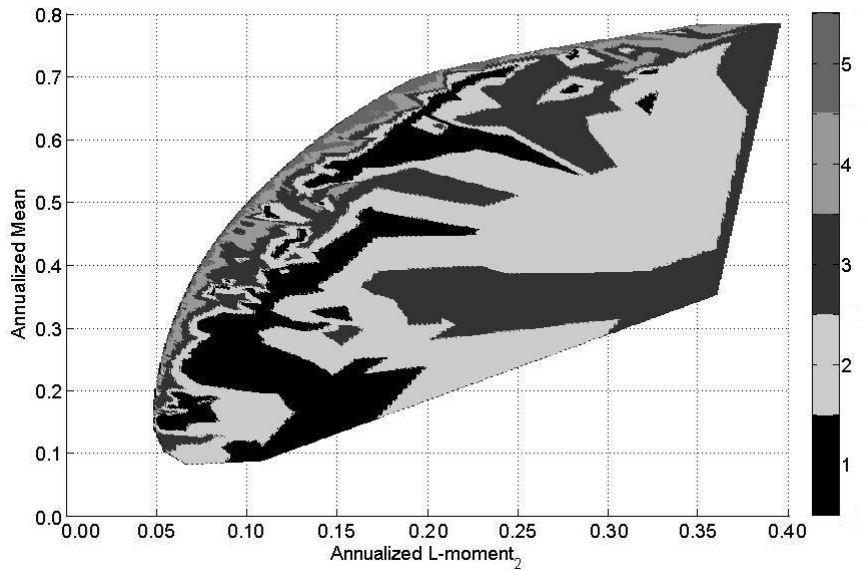


Figure 10. Clusters of the Four L-moment Optimal Portfolios according to their L-moments in the L_1 - L_2 Plane. Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. The colours represent the clusters of the Four L-moment Efficient Portfolios according to their L-moment similarity. This figure is obtained using a Self-Organizing Maps' classification algorithm on a string of five cells, applied on vectors of (unscaled) L-moments defining the optimal portfolios (see Kohonen (2000), and Appendix 3).

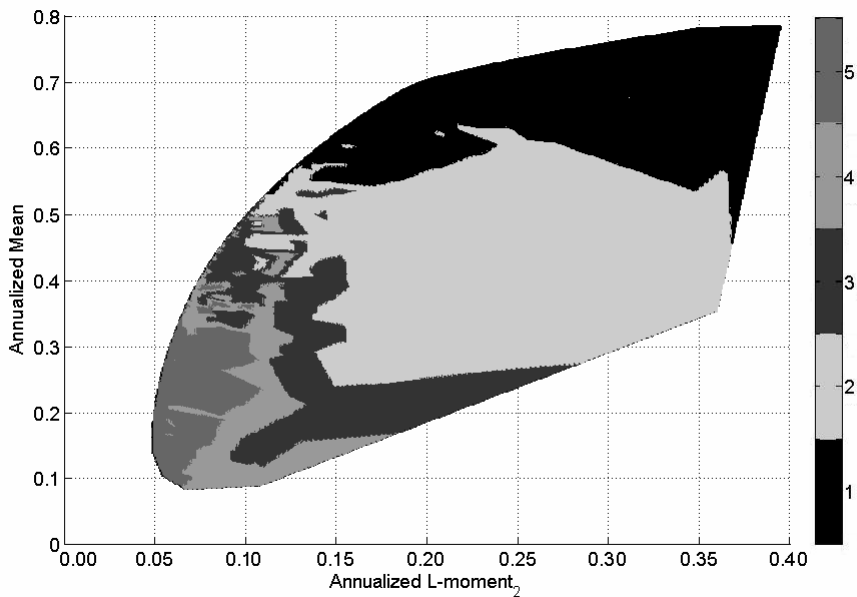


Figure 11. Clusters of the Four L-moment Optimal Portfolios according to their L-moment Rankings in the L_1 - L_2 Plane. Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. The colours represent the clusters of the Four L-moment Efficient Portfolios according to their L-moment ranking similarity. This figure is obtained using a Self-Organizing Maps' classification algorithm on a string of five cells, applied on vectors of L-moment empirical quantiles related to optimal portfolios (see Kohonen (2000), and Appendix 3).

**CRRA Utility Function
with L-moments**

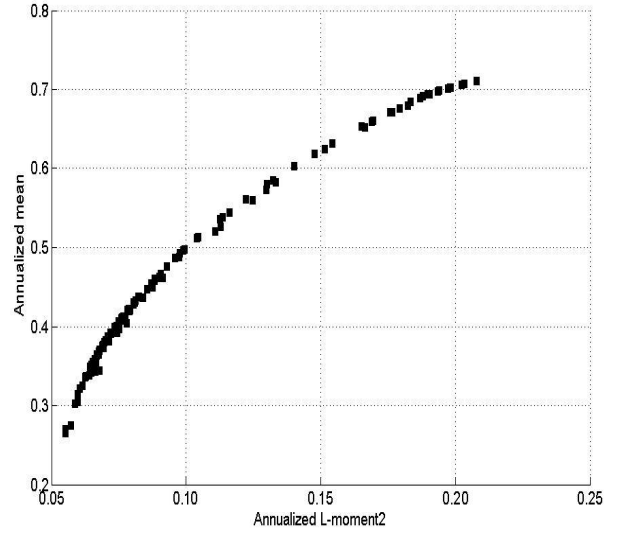
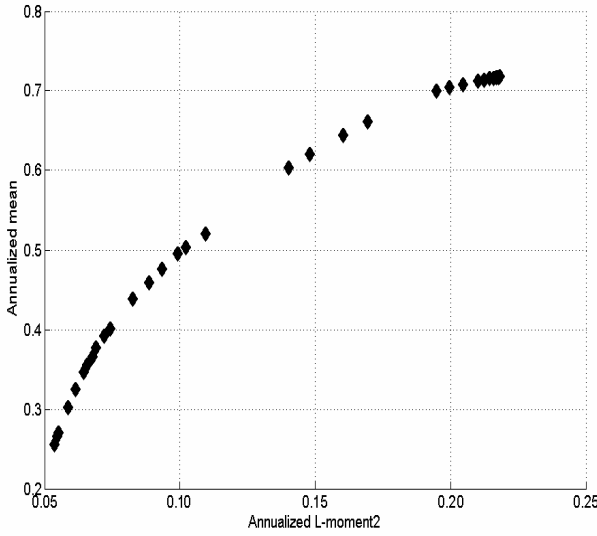
$$U(W) = (1 - \gamma)^{-1} W^{1-\gamma}$$

with $\gamma \in]0, 50[$.

**Quartic Utility Function
with L-moments**

$$U(W) = \beta_0 + \beta_1 W - \beta_2 W^2 + \beta_3 W^3 - \beta_4 W^4$$

with $\beta_i \in [0, 1]$ for $i = [1, \dots, 4]$ and $\beta_i > \beta_{i+1}$ for $i = [1, \dots, 3]$.



**Power-expo Utility Function
with Conventional Moments**

$$U(W) = \alpha^{-1} [1 - \exp(-\alpha W^{1-\gamma})]$$

with $\alpha \in [-50, 50] \setminus \{0\}$, $\gamma \in [-10, 10] \setminus \{1\}$ and $\alpha(1-\gamma) > 0$

**Power-expo Utility Function
with L-moments**

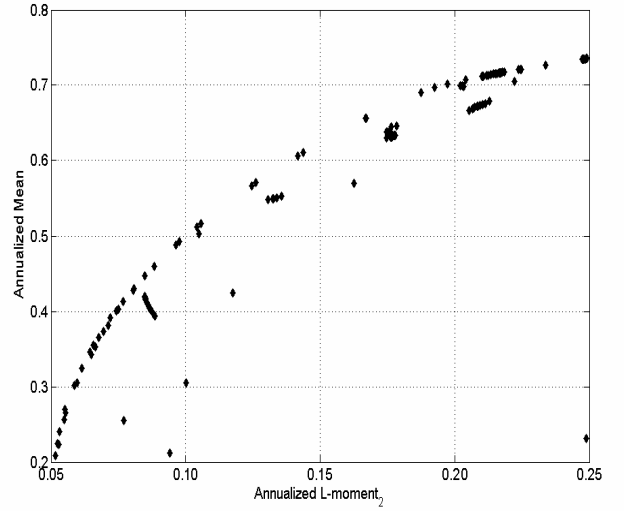
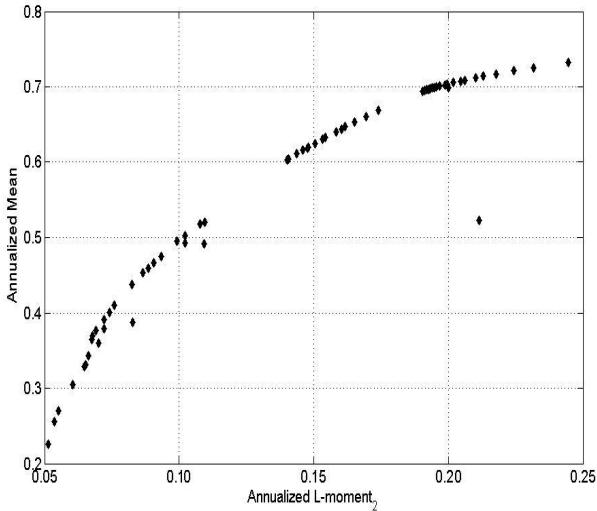


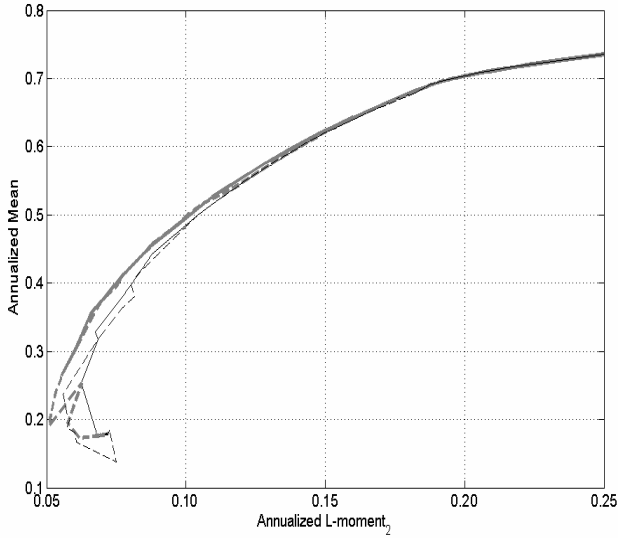
Figure 12. Optimal Portfolios for various Four-moment Dependent Utility Functions in the L_1 - L_2 Plane. Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. From the set of directionally optimized portfolios, we select portfolios that are optimal for each of the considered utility functions. In the top left figure, black points represent optimal portfolios when considering CRRA utility functions. In the top right figure, black points represent optimal portfolios associated to Quartic utility functions that respect the condition of decreasing order moment preferences. In the bottom figures, black points represent optimal portfolios with respect to Power-expo utility functions using conventional moments (bottom left figure) and L-moments (bottom right). Consideration of both C-moments and L-moments allows us to ensure that the two frontiers remain in line and to show the impact of the use of robust statistics in portfolio choice.

CRRA Utility Function with L-moments

$$U(W) = (1 - \gamma)^{-1} W^{1-\gamma}$$

with $\gamma \in]0,50]$.

with Intensified L_3 - (x2) and L_4 - (/2) moment Features



with Intensified L_3 - (x4) and L_4 - (/4) moment Features

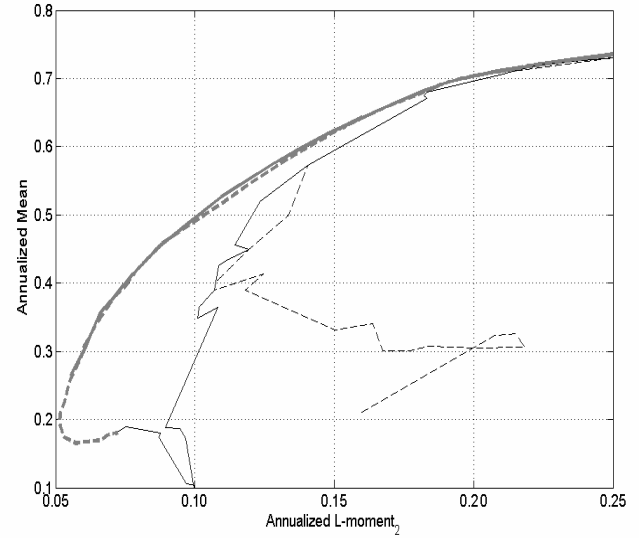


Figure 13. Efficient Frontiers for CRRA Utility Functions with Intensified Higher-order Moments. Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. From the set of directionally optimized portfolios, we select those that are optimal for one of the various CRRA utility functions. On the left (right) figure, the grey bold line represents the CRRA efficient frontier with no change (original data), whilst the thin black line represents the efficient frontier when the third moments of underlying portfolios are multiplied by 2 (respectively by 4), the bold grey dotted line represents the efficient frontier when the fourth moments are divided³⁷ by 2 (by 4) and the thin black dotted line represents efficient frontier when third moments are multiplied by 2 (by 4) and fourth moments are divided by 2 (by 4).

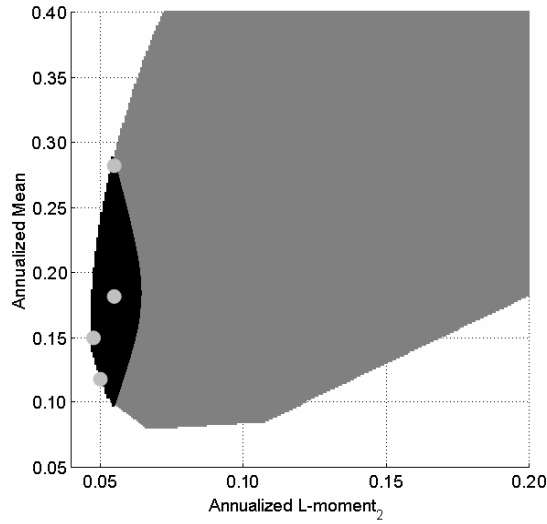


Figure 14. Set of Portfolios with Second L-moments almost Equivalent to the one of the Minimum L-moment₂ Portfolio. Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. The black area represents the region in which the Four L-moment Efficient Portfolios have an L-moment₂ not significantly different at a 5% threshold from the L-moment₂ of the Minimum L-moment₂ (using the Log-normal volatility hypothesis by Alizadeh, Brandt and Diebold (2002) - differences in the Second L-moment are in the third digit). Grey dots represent (from the highest mean to the lowest) the Local Maximum Mean portfolio, the Local Minimum L-moment₄, the Local Minimum L-moment₂, and the Local Maximum L-moment₃.

Table I
P-statistics of Normality Goodness-of-Fit Tests of the various *Optimal* Portfolio Return Series
(frequencies of rejections at 1%, 5% and 10% confidence levels)

Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. From the sets of the (162) original return series, the (65,253) general Four L-moment Efficient Portfolios (see Figure 8), the (379) Four L-moment CRRA Optimal Portfolios, the (123) Four L-moment Quartic optimal portfolio, the (92) Four C-moment Power-expo optimal portfolios, the (341) Four L-moment Power-expo Optimal Portfolios (see Figure 11), we compute the Jarque-Bera, the Kolmogorov-Smirnov, the Liliefors and the Anderson-Darling Goodness-of-Fit tests. The table reports the frequency of P-statistics below the usual probability thresholds (1% between parentheses, 5% between brackets and 10% between accolades), *i.e.* the relative number of (stock) portfolio series being non-normal in the sense of the usual Gaussianity tests.

	Jarque-Bera	Kolmogorov-Smirnov	Liliefors	Anderson-Darling
Original European Stock Returns Series	(.91)/[.96]/{.98}	(.20)/[.38]/{.58}	(.68)/[.81]/{.86}	(.82)/[.90]/{.94}
Four L-moment General Efficient Portfolios	(.65)/[.72]/{.75}	(.15)/[.38]/{.50}	(.64)/[.82]/{.89}	(.78)/[.91]/{.94}
Four L-moment Quartic Optimal Portfolios	(.88)/[.93]/{.95}	(.13)/[.35]/{.50}	(.69)/[.80]/{.84}	(.74)/[.81]/{.83}
Four L-moment CRRA Optimal Portfolios	(.69)/[.78]/{.82}	(.12)/[.15]/{.18}	(.20)/[.26]/{.45}	(.46)/[.48]/{.56}
Four C-moment Power-expo Optimal Portfolios	(.98)/[.99]/{1.00}	(.10)/[.23]/{.43}	(.60)/[.79]/{.84}	(.78)/[.85]/{.86}
Four L-moment Power-expo Optimal Portfolios	(.83)/[.95]/{1.00}	(.14)/[.39]/{.44}	(.55)/[.60]/{.63}	(.65)/[.76]/{.78}

Table II
Statistical Properties of Minimum L-moment₂ Portfolio *versus* the Local Maximum Mean, Local Maximum L-moment₃ and Local Minimum L-moment₄ Portfolios

Source: *Bloomberg*, Weekly Net Asset Values (06/2001-06/2006) in EUR; computations by the authors. From the set of Four L-moment Efficient Portfolios with a L-moment₂ that is not significantly different from the one of the Minimum L-moment₂ portfolio (at a 5% threshold), we select the Local Maximum Mean, Local Maximum L-moment₃ and Local Minimum L-moment₄ Portfolios (see Figure 14). The table presents the first four L-moments (the two first L-moments being annualized and the two last ones being divided by 1,000), and Sharpe ratios of selected local best portfolios.

	Mean	L-moment ₂	L-moment ₃	L-moment ₄	Sharpe ratio
Minimum L-moment₂	.15	.05	-.70	1.30	1.19
Local Maximum Mean	.28	.06	-.90	1.40	2.34
Local Maximum L-moment₃	.11	.05	.20	1.70	.71
Local Minimum L-moment₄	.17	.06	-1.10	.60	1.29